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Abstract

Struggle and its connection to learning are central to improve student learning and understanding of mathematics. A description of what a student's productive struggle looks like in the setting of classrooms can provide insight into how teaching can support or hinder the student's learning process. In order for any struggle to be productive, these struggles with mathematics must be documented. However, prior studies on student struggles are limited and have primarily focused on examining whether or not struggle occurred. Thus, the purpose of this study is to describe pre-service middle grade teachers' (PSTs) struggle types in details as well as to investigate how enagaging in a non-routine high level task (doing mathematics) fosters (or inhibits) productive struggle during instruction.

Key words: Geometry; Productive struggle; Pre-service teachers

Introduction

Hiebert and Grouws (2007) definde struggle as an intellectual effort students expend to make sense of mathematical concepts that are challenging but fall within the students' reasonable capabilities. Struggle often conveys negative meaning and is viewed as a problem in mathematics classrooms (Hiebert& Wearne, 2003; Borasi, 1996; Sherman, Richardson, & Yard, 2009). Researchers, however, suggest that struggling to make sense of mathematics should be a necessary component of learning mathematics with understanding (Hiebert & Grouws, 2007). In the Common Core Standards for Mathematical Practice (CCSSM), which was considered to be the most robust set of standards adopted widely in the United States (Marrongelle, Szatjn & Smith, 2013), the first standard states that students should "make sense of problems and persevere in solving them" (CCSSI, 2010). Kapur (2010) argues that designing for persistence when students encounter difficulty while working on challenging tasks is central to productive failure and has important implications for effective learning. Teaching that provides students opportunities to struggle with important mathematical ideas has been identified in mathematics education research as one of the key components of teaching that supports the development of students' conceptual understanding of mathematics (Hiebert & Grouws, 2007; Hiebert & Wearne, 1993;Stein, Grover, & Henningsen, 1996; Borasi, 1996).Thus, perseverance or continuing forward irrespective of struggle should be seen as an opportunity for students to grapple with important mathematical ideas.

Struggle and its connection to learning are central to the issue of how to improve student learning and understanding of mathematics (Hiebert & Grouws, 2007). A description of what a student's productive struggle looks like in the setting of classrooms can provide insight into how aspects of teaching can support rather than hinder this instructional process (Kilpatrick et al. 2001; Hiebert & Grouws 2007). However, what struggle looks like in mathematics classrooms and how it can be productive needs further exploration (Warshauer, 2014). Prior research on student struggles has been limited and has primarily focused on examining whether struggle occured without examining in detail the nature of students' struggles (e.g., Inagaki, Hatano, & Morita, 1998; Santagata, 2005). To make up this shortage, Warshauer investigated the nature of productive struggle in middle grade classrooms in details (Warshauer, 2011; 2014a). Warshauer (2014a) states that for future studies there is need to build on the descriptions of struggle in different classroom settingsand to what extent task selection contributes or hindersstudents' productive struggle. This study aims to address this need by investigating the nature of productive struggle in a college classroomand how the selection of tasks used during the class supports or hinders productive struggle.

Thus, the aim of this study is two fold. Firstly, it is to describe pre-service middle grade teachers' (PSTs) struggle types in details. Then, it is to investigate how enagaging in a non-routine high leveltask (doing mathematics) fosters or inhibits productive struggle during instruction. In other words, this study aims to

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describe what struggle looks like in a college classroom for pre-service mathematics teachers and how to support productive struggle in a classroom setting. In order to investigate what productive struggle looks like in a college classroom, two frameworks guided this study, which will be described next.

Frameworks

Productive Struggle Framework

Warshauer (2014a) argued that struggle can be observable in most classrooms even if it is perceived as a phenemenon. Warshauer (2011) developed a productive struggle framework which includes a classification of middle grade students' struggles with description (see Table 1). Warshauer (2011) identified four types of struggles that students encounter as they work on challenging tasks in a classroom session. Students encounter difficulty in figuring out how to get started or carry out their task. They are unable to piece together and explain their emerging ideas, or express an error or misconception in problem solving (Warshauer, 2014b). Even though Warshauer developed these struggle types in a middle grade math classroom, her classification of student struggle types was hypothesized to be similar in a college classroom. Thus, these struggle types served as a framework to analyze the episode of a classroom episode where PSTs were engaged in a challenging non-routine problem to struggle.

Table 1. Productive struggle types

Kind of Struggles	Description
Get Started	• Confusion about what the task is asking • Claim forgetting type of problem • Gesture uncertainty and resignation • No work on paper
Carry out a process	• Encounter an impasse • Unable to implement a process from a formulated representation • Unable to implement a process due to its algebraic nature • Unable to carry out an algorithm • Forget facts or formula
Uncertainty in explaining and sense-making	• Difficulty in explaining their work • Express uncertainty • Unclear about reasons for their choice of strategy • Unable to make sense of their work
Express misconception and errors	• Misconception related to area and perimeter • Misconceptions related to geometric shapes

Levels of Cognitive Demand of Tasks

There could be various opportunities for students to struggle that can be productive in understanding mathematics. One example of students' struggle that can be productive in learning mathematics could be integrating challenging problems (Hiebert & Wearne, 2003). Mathematical tasks, in particular those that place high level of cognitive demands including making connections among concepts and sharing, explaining, and justifying one's solution (Boston & Smith, 2009; Hiebert, Carpenter, Fennema, et al, 1996; Ball, 1993, Doyle, 1988), provide a classroom context for students to develop their conceptual knowledge and understanding (Hatano, 1988, Hiebert, 1986; Zaslavsky, 2005; Goldman, 2009; Fawcett & Gourton, 2005).

Tasks are a central part of a teacher's instructional activities, and what students learn is often defined by the tasks they are given (Christiansen & Walther 1986). In order to move students toward developing a deep conceptual understanding of mathematics, classroom teaching must incorporate opportunities for students to grapple with meaningful tasks (Lampert 2001; NCTM 1991; Schoenfeld 1994). Stein et al., (1996) identified four levels of cognitive demand that are hierarchical. From lowest to highest they are: *memorization, procedures* *without connections, procedures with connections, and doing mathematics*. The characteristics of each level are described below. These characteristics of levels are used to define the cognitive level of the task used during the instruction in this study. How these levels guided the coding of the task is going to be described later in details in the methodology section.

Table 2. Levels of cognitive demands of tasks

Levels of Cognitive Demand	Characteristics of Each Level
Memorization	• involves either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas, or definitions to memory • involves very similar reproduction of previously seen material
Procedures without connections	• are algorithmic • have no connection to the concepts or the meaning that underlie the procedures being used • are focused on producing correct answers rather than developing mathematical understanding, explanation, or justification
Procedures with connections	• focus use of procedures for purposes of developing deeper levels of understanding of mathematical concepts and meaning • usually represented in multiple ways with connections among multiple representations • suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms with concepts that are not transparent • engage with conceptual ideas that underlie the procedure to complete the task successfully
Doing mathematics	• requires complex and non-algorithmic thinking • requires exploration and understanding the nature of mathematical concepts, processes, or relationships • demands self-monitoring or self-regulation of one's own cognitive processes • requires access to relevant knowledge and experiences and make appropriate use of them • requires analysis of task and examine task constraints that may limit possible solution strategies and solutions • requires considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution processes that are required

Method

Participants

Participants were 48 pre-service middle grade teachers who are going to teach middle grade mathematics (grade5-8) after they complete their teacher education program at a public university in Turkey. The participants are in their third year of the program. They completed a required Geometry course in their first year and they all passed with a passing grade C+ or above prior to the study.

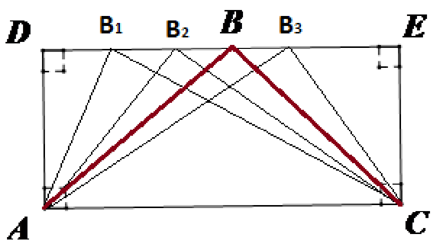
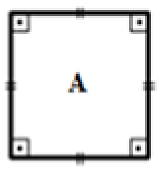
Data Sources

In order to investigate how PSTs, engage in a non-routine task and how they overcome their productive struggle, data sources included semi-structured group interviews with 16 PSTs and an episode of a class session where PSTs were engaged in solving a non-routine high level task (see figure 1).

Semi-structured Interviews

The participants were interviewed as groups of four for about 30-45 minutes by the author. The interview questions aimed to investigate PSTs' conceptions of two underlying concepts that are necessary to complete the task successfully. These concepts are: (1) quadrilaterals and inclusive relationships among quadrilaterals and (2) area-perimeter and the relationship between area and perimeter. Table 3 below shows some sample questions used during the group interviews.

Table 3. Sample interview questions

Sample Interview Questions	
Sample Question 1: Ömer thinks that the area of triangle ABC will not change regardless of where B would be on side DE. Do you think Ömer is right? How do you prove that Ömer's claim is true or false?	Sample Question 3: Ahmet claimed that he found a rule in math class: "if perimeter of a rectangle increases, then its area will increase". Do you think Ahmet is right? How do you prove that Ahmet's claim is true or false?
	
Sample Question 2: Is it possible to have a rectangle that is not a parallelogram? If so, can you draw a picture?	Sample Question 4: A student I talked with earlier told me that the shape below is a rectangle. Do you agree? 

The Task Implementation

The task (see fig. 1), which was used during the class episode, was an adaptation of the task on Figure This! (NCTM, 2004). It was coded as doing mathematics according to Stein et al. (1996) levels of cognitive demands of tasks framework due to several reasons as follows:

It required complex and non-algorithmic thinking. Even though concepts of area and perimeter fall within the PSTs' reasonable capabilities given that they all took high school geometry and a geometry course at college prior to this study, solving this task required much more than knowing how to calculate perimeter and area of various geometric shapes.

It required exploration and understanding the nature of mathematical concepts, processes, or relationships. The task was not a familiar task to the participants, which means that the participants did not solve similar tasks in their college geometry course before. Additionally, the task could be solved in so many different ways, which required an exploration and it allowed making sense of the relationship between the underlying concepts--area and perimeter.

It required access to relevant knowledge and experiences and makes appropriate use of them. Adopting a standard algorithm such as formulas of area and perimeter of geometric shapes could not simply solve the task. However, solving the task required access to that relevant knowledge.

It required analysis of task and examines task constraints that may limit possible solution strategies and solutions. The task required PSTs to construct a solution and justify their solution so that they were asked specifically to evaluate their solutions. The task also had another challenge for the PSTs by asking them to solve the problem with another case (dividing the cake among 6 people), which encouraged PSTs to think deeper specifically about whether their previous solution could be generalizable to other cases.

Due to above characteristics of the task as well as requiring considerable cognitive effort, the task was coded as doing mathematics. The PSTs worked on the task in their groups of 4-6 for about an hour and thirty minutes. The entire time during the PSTs worked on the task was videotaped by a mobile camera in order to capture the moments when the PSTs struggled during instruction. The PSTs were provided graph papers, dotted papers and geobards to work on the task during instruction.

CAKE PROBLEM

Mete invited his best friends to celebrate his birthday. Mete and his two friends want to share rectanglecake, which has 81 square inches surface area on top. It is a chocolate cake with butter cream frosting. The cake is frosted evenly on the four sides and the top. How can Mete cut the cake so that each person receives an equal share of both cake and frosting? How can you justify that each person got the equal amount of cake and the frosting?

- Mete's three other friends come to the party right before cake cutting. How can Mete cut the cake evenly among six people now? How do you know it is fair?

Figure 1. The cake task

Data Analysis

The group interwiews were transcribed and transcripts of the interviews were coded using the open-coding process (Strauss and Corbin 1990). Open coding refers to the process of generating initial concepts from data while theoretical coding conceptualizes how the substantive codes may relate to each other as hypothesis to be integrated into theory (Glaser & Strauss, 1967). Thus, open coding involves identifying, naming, categorizing and describing phenomena found in a text line by line. The interview transcripts and participants' written responses were carefully read, initial impressions summarized, and interesting issues regarding participants' conceptions of polygons and area-perimeter highlighted. Then, the transcripts were coded line-by-line by using a constant comparative method of coding as proposed by Glaser and Strauss (1967).

After PSTs' group interviews were analyzed, all the parts during classroom episode where PSTs made mistakes, expressed misconceptions, or claimed to be confused, and to which instructor responded were identified. Using an embedded case study methodology (Yin, 2009) with instructional episodes as the unit of analysis, I identified and described the nature of the students' struggle by usingWarshauer's productive struggle types (see table 1). Additionally, I recognized the instructional practices of the instructor that either supported and guided or did not support or guide the students' sense- and meaning-making of the mathematics in the classroom episodes. As an exploratory case study, the goal of my data analysis was to identify, describe and examine the struggles PSTs encountered during their engagement with the task.

Results and Discussion

In this part, the findings of the study will be described in details.

Interview Results

Semi structured interviews were used to investiagte PSTs' conceptions of two concepts—area-perimeter and polygons. PSTs' responses to the interview questions revealed their conceptions as well as misconceptions regarding these two concepts. In this part, their responses to the interview questions will be demonstrated.

PSTs' responses to the questions about polygons demosntrated that PSTs struggled with the concept of polygons, specifically defining polygons and identifying inclusive relationships among geometric shapes. That is, being able to classify one quadrilateral as another (i.e. a square is a rectangle).

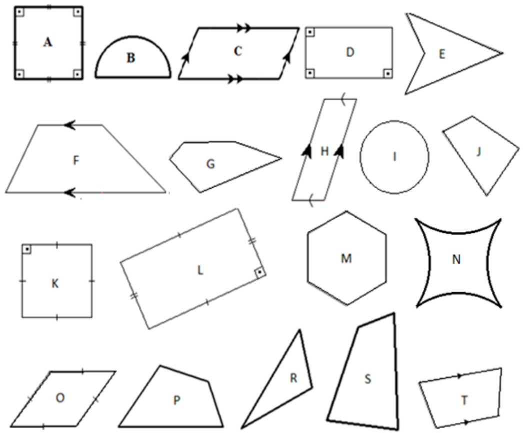


Figure 2. Geometric shapes used during interviews

The PSTs who participated in the group interviews struggled to provide proper definitions for polygons and quadrilaterals. As can be seen in the excerpt below, PSTs struggled to group the shapes in figure 2 above into two groups. One PST attempted to group the shapes as polygons versus non-polgons but struggled to define his sorting criterion.

1. Interviewer: Can you sort these shapes (referring to the shapes in figure2)into two groups?
2. PSTA: I think they form more than two groups.
3. Interviewer: Can you make only two groups?
4. PSTA: No.
5. Interviewer: I think you sort them into two groups (referring to one of the PSTs). Can you tell us what you did?
6. PSTB: When I look at these shapes, I see regular shapes and um, what are these called? Circular shapes?(referring to the shapes like B, N or I)
7. Intervewer: Non-regular shapes?
8. PSTB: Yeah!
9. Interviewer: Ok, which ones are non regular shapes?
10. PSTB: B, N, I
11. Interviewer: What do you mean by regular and non regular shapes?
12. PSTB: Regular shapes consist of line segments, but non regular shapes do not consist of line segments.

It was evident that PSTB was grouping the shapes according to the criterion of having straight sides versus not having straight sides. In other words, he was grouping the shapes into two groups as polygons versus non-polygons. However, he was not aware that this sorting criterion was indeed a necessary criterion of being a polygon which will come out later during the interview. Fujita and Jones (2006) claim that pre-service teachers' subject knowledge of geometry is amonst their weakest knowledge of mathematics. It was evident in the excerpt above that the PSTs were struggling to identfying basic geometric shapes. Even though pre-service teachers are expected to have solid knowledge of mathematics that they are going to teach, studies have documented that pre-service teachers share many of the same struggles and hold similar misconceptions (Fujita & Jones, 2006; 2007; Zilkova, 2015) with students (de Villiers, 1994; Fujita, 2012; Jones, 2000; van Hiele, 1999).

In the excerpt below, the PSTs were engaged in a discussion about what a quadrilateral is. As they struggled to define polygons, they also struggled to describe a quadrilateral. It was evident in the excerpt below that PSTA struggled to identify what a quadrilateral is. She argued that shape N could be considered as a quadrilateral. However, PSTB was able to recognize that it could not be a quadrilateral since its interior angles were not formed by straight sides. Yet, he got confused whether shape N could still be considered a polgon even if it was not a quadrilateral.

1. Interviewer: Lets look at shape N. Do you think it is a quadrilateral?
2. PSTC: Yes, because it also has 4 sides, it is closed and it has 4 corners. I think it is a quadrilateral!
3. PSTA: I am not sure! Because I do not know whether these could be interior angles (referring to the space between the two sides of shape N). That's why I am not sure.
4. PSTB: I think these are not angles because angles should be between two lines. If you wanna measure these angles, if we call this an angle, it would give different measurement at different points so thats why angles should be formed between two straight lines.
5. Interviewer: So, shape N is not a quadrilateral because it does not have interior angles for you (PSTB) and it is not a polygon either?
6. PSTB. It is not a quadrilateral but it can be a polygon.
7. Interviewer: So, what is a polygon?
8. PSTB: It should be a closed figure
9. Interviewer. Can you show me a polygon on this figure?
10. PSTB: There are a lot. For instance, B,I
11. PSTA: They are circles
12. Interviewer: Are circles polygons?
13. PSTB: Of course they are!
14. PSTA. Polygons should have at least 4 sides
15. Interviewer: How many sides does shape "I" have?
16. PSTA: None!
17. PSTs (laughs).

In addition to engaging PSTs in the discussion of defining polygons, they were also engaged in a discussion about whether a square could be considered as a rectangle (see sample question 4 in table 3) in order to investigate their conceptions of inclusive relationships among quadrilaterals.

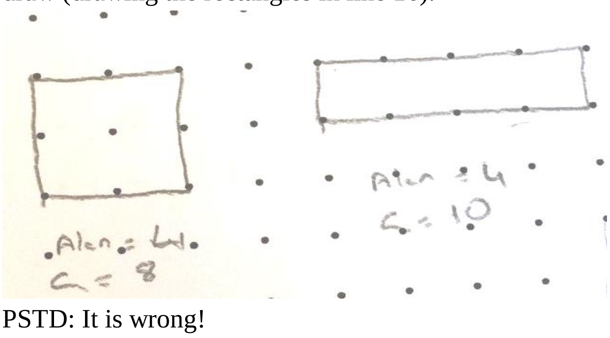
1. Interviewer: The PSTs I interviewed before argued that shape A (pointing the shape on fig 2) is a rectangle. Do you agree with them?
2. PSTA: No!
3. PSTC: No, it can not be a rectangle.
4. PSTB: It can be. It has right angles.
5. PSTA: But all sides are equal. A rectangle has opposite sides equal.
6. PSTB: Opposite sides are equal in this one too.
7. Interview: ok, so you disagree with the claim (referring to PSTA and PSTC)?
8. PSTs: Yes
9. Interviewer: But you think that they might be right? (Referring to PSTB)
10. PSTB: Yes, becuase this shape has all the things that a rectangle has.

In geometry and spatial sense standards, it is important that students grasp connections between geometric concepts (CCSSI,2010; NCTM, 2000). Leung (2008) argues that learning to identify geometric shapes and understand inclusive and transitive preperties among these shapes is prerequsite for learning further concepts such as set theory or conguerence and similarity. A number of studies, however, have reported on students' difficulties with the hierarchical classification of quadrilaterals (Fujita & Jones 2007; van Hiele, 1999). In order for students to grasp this connection among geometric shapes as highlighted in the standards, the teachers themselves should be competent in identifying and describing inclusive relationships among shapes first. However as it was evident in this study, the PSTs who are just a few steps away from being a teacher were struggling with these ideas.

Even though PSTs struggled with the questions about polygons including defining polygons and identfying inclusive relationships among polygons as can be seen above, they were more comfortable with the questions regarding to the concepts of perimeter and area.

1. Interviewer: Can you find the perimeters of these two rectangles?
2. PSTs (Calculating the perimeter by adding the length of the two sides and multiplying by two or counting the square units.)
3. Intervewer: So, what is the perimeter?
4. PSTs: 20 and 26
5. Interviewer: How did you find?
6. PSTs: by adding the side lengths or counting the sqaure units.
7. Interviewer: If you ask this question to 6th graders, how would they respond?
8. PSTD: They would count the units
9. Interviewer: Can you show me how?
10. PSTD: They need to count all squares once but the corner squares twice
11. Interviewer: Why?
12. PSTD: Becuase we use two sides of these squares.

It was evident in the excerpt above that PSTs were able to calculate the area and perimeter by using the standard formula as well as by counting the units. Additionally, PSTs were able to recognize that the corner tiles should be counted twice while calculating the perimeter since they contibuted to both sides. Eventhough the PSTs were confortable calculating the area and the perimeter of the presented rectangles in this study, there is evidence in the literature that teachersstruggle with the concepts of perimeter and area (Baturo & Nason, 1996; Fuller, 1997; Heaton, 1992; Ma, 1999). They show limited understanding and various misconceptions such as believing that there is a constant relationship between area and perimeter (Baturo & Nason, 1996; Simon & Blume, 1994; Ma, 1999). Below is an excerpt to demonstrate PSTs' responses to the sample question 3 in table 3.

13. Interviewer: Ahmet found a rule in math class: if perimeter of a rectangle increases then its area will increase. Do you agree with Ahmet?
14. PSTs: Yes, umm (PSTs are drawing examples of rectangles to check the claim)
15. PSTD: I think it is wrong. Because the perimeters would be lets say 8 and 10 but, umm, let me draw (drawing the rectangles in line 16).
16. 

17. PSTD: It is wrong!
18. Interviewer: So the perimeter could increase but the area might stay the same.
19. PSTD: Yes.

Stavy and Tirosh (2000) describes an intuitive rule—More A- More B—, which evolves from experiences in everyday life. In this intuitive rule, a perceptual quantity (A) can serve as a criterion for comparing another quantity (B). Stavy and Tirosh (2000) argue that this rule is common core to many misconceptions in science as well as in mathematics. In this case, PSTs questionedwhether the perceptual quantity –perimeter– is bigger, then another quantity –area–would also be bigger at first. Later, they were able to construct an example that contradicted to the claim.

PSTs' Struggle with The Task

During the class episode where PSTs were engaged in working on the task in their groups of 4-6, they were provided with graph paper, dotted paper and geoboards to choose from in order to work on the task. In this part, how PSTs worked on the tasks in their groups, what types of struggles they encountered during this process and how they overcome their struggle will be described. In other words, the nature of struggles that the PSTs had during the task implementation and the instructor's responses to these struggles will be decribed. Kapur (2010) argues that as students engage with a task, they must reach an impass in order to learn. Otherwise, if students did not reach an impasse, learning was rare (Kapur, 2010). Thus, the types of struggles that the PSTs encountered as well as the kinds of responses that the instructor provided to these struggles sould be documented. In this section, I will describe each struggle in greater detail.

Express Misconceptions

As it was evident in the group interviews that PSTs who participated in the interviews demonstrated misconceptions specifically related to defining polygons and identifying inclusive relationships among quadrilaterals. Therefore, it was anticipated that some of those misconceptions come out and might serve as a hindrance during the instruction. When the task was posed, some of the PSTs attempted to find rectangles that are not squares with the area of 81 since they believed that the cake should be a rectangle not a square.

1. PST: I don't understand the task. If the area on top of the cake was 81, then it can be 3 by 27 or 9 by 9, because 27 x 3 is 81 and 9x9 is 81. But it says a rectangle cake, so it can not be 9 by 9?
2. Instructor: Why it cannot be 9 by 9?
3. PST: Then, it would be a square.
4. Instructor: Hmm, so it should not be a square?
5. Another PST: It can be a square. Square is a rectangle.
6. Instructor: (To the whole class). I have heard many of you ask the same questions about the task. Lets talk together for a second. If the top of the cake has the area of 81, then what would the cake look like?
7. PSTs: 3 by 27 or 9 by 9.
8. Instructor: Is that it? Can we draw another rectangle that has an area of 81 square inches?
9. PSTs: No
10. PST: Yes, if the sides can be different than whole numbers. We can do, umm, (using a calculator) 6 by 13,5.
11. Instructor: Okay. We can draw different rectangles then. Not just 2! But, what about 9 by 9 square? Can we draw a square cake even if the task says a rectangle cake?
12. PSTs: Yes
13. PSTs: No
14. Instructor: I have heard yes and no
15. PSTs: A square is a rectangle!

As can be seen in the excerpt above, some of the PSTs struggled to accept a square as a rectangle. This limited knowledge of quadrilaterals served as a hindrance for some of the PSTs to proceed with task. The instructor chose to probe PSTs' thinking by moving the discussion of whether the cake could look like 9 by 9 square to the whole class in order to address PSTs' different attempts and in order to build on their ideas.

Warshauer (2014a) described this kind of responses as *Probing Guidance*. Hiebert and Stigler (2004) define such support and guidance as "provide students with opportunities to think more deeply about mathematical concepts" (p. 13). The instructor used probing guidance as a response to PSTs' struggle regarding to whether the cake could be a square in order to make PSTs' thinking visible and to serve as the basis for addressing their struggles. Schleppenbach, Flevares, Sims, and Perry (2007) emphasized the role that student mistakes can play in promoting discourse about mathematical topics and in assessing student understanding of mathematical topics.

Get Started

Warshauer (2014a) argues that this type of struggle—*Getting Started*— occurres as students are attempting tasks of higher-level cognitive demand. When PSTs were introduced to the task at the beginning of the class, they voiced confusion about what the task was asking, gestured uncertainity and hesitated to put anything on paper, which implied that PSTs struggled to get start on solving the task.

1. PST: How many shares should we do?
2. Instrutor: Mete invited his two best friends, so how many pieces should they have all together?
3. PSTs: Three
4. Instructor: You should divide the cake up into three pieces. But you have to be careful that eveyone wants to have an equal amount of the cake and the frosting.

As can be seen in the excerpt above, PSTs asked questions such as what the task was asking in order to make sense and get started working on the task. The instructor chose to simply answer these questions of the PSTs which was defined as *Telling* by Warshauer (2014a).

1. PSTA: It says 81 square inches, so how do we draw?
2. Instructor: If the area on top was 81, then you said it could be 3 by 27 or 9 by 9 rectangles. You need to draw the cake first and then try to divide it up among three people like him (referring another PST sitting right next to).

In the excerpts above, PSTA struggled to start working on the task and tried to resolve his struggle of getting strated on the task by seeking guidance from the instructor. As aresponse to his question, the instructor chose to directly guide the PST by suggesting to start with drawing the cake similar to the one that was drawn by another PST at the table, which was referred to as *Directed Guidence* by Warshauer (2014a) along with highlighting the important aspect of the task. Directed guidance responses appeared to redirect student thinking toward the teacher's thinking, narrowed down possibilities for action, directed an action, broke down problems into smaller parts or altered problems to an analogous one such as from an algebraic to a numerical (Warshauer, 2014a).

Carry Out a Process

Struggles to carry out a process are indicative of the difficulty students have connecting procedure to concepts (Warshauer, 2015). As Warshauer described, struggles to carry out a process in this study were indicative of PSTs' difficulty to connect their procedural knowledge of calculating area and perimeter to more conceptual knowledge of using the concepts of area and perimeter to solve a non-routine task. Boaler (1998) states that being able to execute a procedure did not guarantee that students could solve a task that involved procedures with connection or doing mathematics. PSTs were able to calculate the perimeter of a square with its given area of 81 square units. Even though they were able to find that each person should receive a piece with 12 units of sides with icing and 27 units of surface area on top, they were not able to use these to construct their solution for the task.

1. PST: The area on top is 81, so each part should have the area of 27. We can do something (drawing the figure below).
2. PST1: The perimeter is 36,so 12 (referring each pieces side lengths with icing)
3. PST: If we do this, umm, no it does not work (referring to he figure..)
4. PST1: Yes it works. See one side and three units (highlighting the sides of one of the parts that is the shape of a trapezoid) and one side and three units (highlighting the sides of the other trapezoid).
5. PST: What about the middle one? There is only one side
6. PST1: No, it has this one too (referring to the sides of three units). One side and three units.
7. PST2: Areas are different.
8. PST1: (Attempted to count the square units). Yes.
9. Instructor: So it does not work?

10. PSTs: No, not for the area
11. Instructor: Each piece shuld have sides of 12 units with icing and top area of 27 unit squares? Can you change your cuts to satisfy this criteria?
12. PSTs: We do not know.

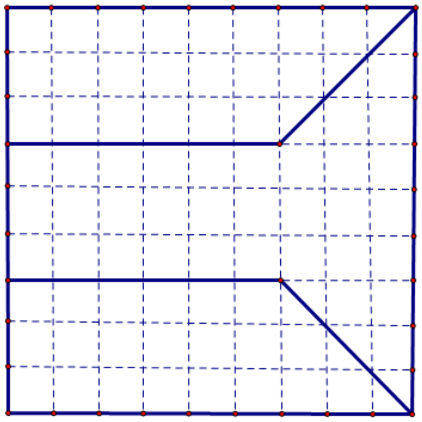


Figure 3. An example of incorrect solutions

As it was evident in the excerpt above, this solution above demonstrated the fact that being able to execute a procedure to find the area and perimeter of geometric shapes did not always guarantee that PSTs could solve a task that involved higher cognitive level tasks. The instructor revoiced what the PSTs were discussing about the requirement of the task (each piece having sides of 12 units with icing and 27 square units of top area) and afforded more time to work on the task. This response of the intsructor was coded *Affordance* according to Warshauer's framework. Affordance type of teacher responses provided opportunities for students to continue to engage in thinking about the problem and build on their ideas with limited intervention by the teacher. Teachers were explicit in encouraging students to continue their efforts in their tasks. The teachers had to monitor the progress of the students, however, as momentum in doing the mathematics was lost at times when the students could not navigate beyond their struggle.

The solution that will be described below was a correct solution which was represented some of the initial solutions that the PSTs constructed. Many PSTs attempted to partition the cake into four squares initially as they saw this as an easier task than partitioning the cake into three equal pieces. This ensured an equal amount of cake with equal amounts of frosting both on top and the sides. Then, they partitioned one of those pieces into three equal sized pieces.

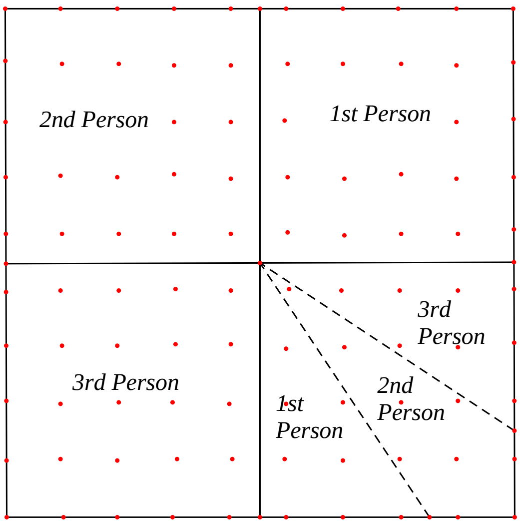


Figure 4. An example of correct solution with multiple pieces

1. PSTs: We can cut the cake into forths instead. Then we divide the forth piece among three people (drawing the figure 4.)

2. Instructor: Are they equal (referring to each piece that a person would recieve)?
3. PST: Squares are equal! The area of this is 3x4,5 and divided by 2 (referring to one of the triangular pieces)
4. Instructor: What is the area of this piece (referring to the part that is a shape of a kite)
5. Another PST: It is not a traingle. How can we find the area of it? (Silence)
6. Instructor: Ok, work on this and check whether all of these second pieces are equal as well.

The group who came up with this solution attempted to justify that each person received the same amount of cake as well as frosting by calculating the area of the pieces and counting the sides with frosting. In order to calculate the area, the PSTs used their knowledge of area formulas for squares and triangles. Although they applied spatial reasoning to identify possible partitions, PSTs tended to resort to the formula and not consider other geomertic approaches; for example, further partitioing the shape into square units (geoboard areas) to determine the area. As such, for those who did not identify the portion for the 2nd person as a kite, they struggled to prove that the three smaller pieces were of equal area.
The instructor again responded to the group which was coded as *Affordance*. The instructor chose to give more time to the PSTs to calculate the area of the small piece that has a shape of kite instead of showing them how to find it.

Uncertainity in Explaning and Sense Making

One of the groups was able to construct a solution as demonstrated below (see fig. 5). The PST who constructed the solution below argued that dividing the cake into thirds by using 120 degree angles, which is 1/3 of 360 degrees, would ensure that each person would recieve an equal amount of the cake and the frosting. However, as can be seen in the excerpt below some of the PSTs at the table needed further justification that this strategy sufficed.

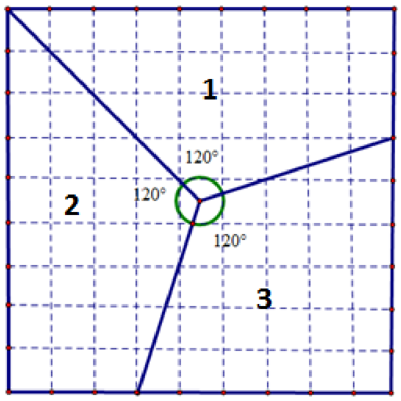


Figure 5. An example of correct solution with three pieces

1. PST: I divided the cake into 120 degrees angles from the center of the square. The cake and the frosting will be equal for three pieces.
2. Another PST: But what about the surface area, would they be equal? Look at this one (referring to the the piece that was labelled as 2), it looks bigger.
3. PST: It should be equal, because we are dividing the cake into three pieces and the angles are equal.
4. Instructor: You claim that if you use 120 degree angles then each piece will be fair?
5. PST: Yes
6. Instructor: Can you show that each piece is indeed equal? What should be the surface area of each piece?
7. PST: 81 divided by 3 is 27. Each part should have the area of 27
8. Instructor: Ok, so the area should be 27. What about the side lengths with frosting?
9. PST: It should be 36 divided by 3, 12. It should be 12.
10. Instructor: So we have two important numbers 12 and 27. Can you check whether each part has the area of 27 and the sides with frosting of 12?

The instructor chose to directly guide the PST who constructed the solution above to calculate the area of each piece since some of the PSTs at the table were not conviced that each piece would have the same surface area. At another table, one of the PSTs suggested a strategy to cut the cake into forths first since it was easier than cutting the cake into thirds. Then, partioning each piece into thirds to make sure to divide the cake equally among three people. Even though his suggestion was not a practical solution, it would indeed work. However, what he sugessted was not accepted by his group members as can be seen in the excerpt below.

1. PST: 3 people are coming to the party. If we divide the cake into 4 and then cut each piece again into three? What do you think?
2. Another PST: Shrugged his shoulder
3. PST: It does not work?

After his suggestion was ignored and not accepted by his group members, he continued to work on his strategy alone. He then constructed the solution below (see figure 6). As can be seen in the figure, he realized that he did not need to partition each of four pieces into three parts. Instead, he cut the cake into forths first and then divide one of the pieces into three equal parts. His solution was coded as a correct solution despite the fact that he partitioned the cake into more than three pieces as all PSTs were encouraged to divide the cake up into only three pieces.

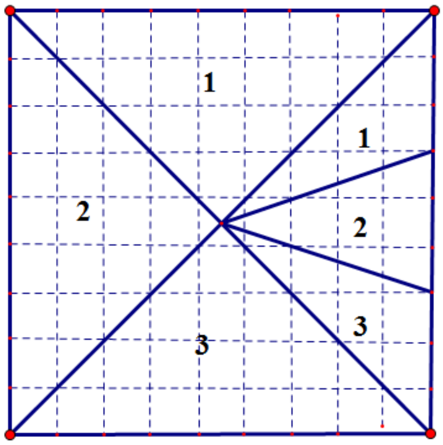


Figure 6. Another example of correct solution with multiple pieces

4. PST: I think this works. These (referring to triangular peieces) have the same base and the same height.
5. PSTs: yeah!

Although his initial idea was not accepted, after he completed his solution his group members agreed that it would work.

Exploring Responses to the Cake Problem

PSTs' notes and solutions were collected at the end of the class in order to investigate how many different solutions were constructed. PSTs' solutions were grouped into three categories as follows:

(1) Category 1: Correct solutions with multiple pieces involves partitioning the cake into multiple pieces (more than three) to satisfy the criterion of cutting the cake. These solutions were initilally constructed by the PSTs since it was easier to cut the cake into multiple pieces as opposed to cutting it into only three pieces.
(2) Category 2: Concrete solutions requires to cut the cake into three pieces only and involves a justification by simply counting the provided units of measure—square units.
(3) Category 3: Abstract solutions involves a solution and a justification that does not require simply counting the provided units of measure, but instead using mathematical rules or constructing different units of measure.

Each of these categories will be described in detail next.

Category 1: Correct Solution with Multiple Pieces

When the task was posed, PSTs initilally attempted to cut the cake into more than three pieces (e.g. cutting the cake into four congruent pieces) since it was easier solution than the original task. Boaler (2008) argues the importance of using real-world problems that are realistic to support students in making sense of problems as well as solutions. The real-world context of the problemhelped PSTs who attempted to cut the cake intomultiple pieces (more than three pieces). For instance PSTs who constructed the solution in figure 7 wereable to seewhy the solution was correct but impractical due to the conext of the task. PSTs were encouraged to find a way to cut the cake into three pieces only so that each person received only one big piece instead of multiple pieces. Asking this question encouraged PSTs to think more deeply about the area and perimeter of theoriginal cake and how it should be distributed among three people, thereby encouraging PSTs to consider even more solutions.

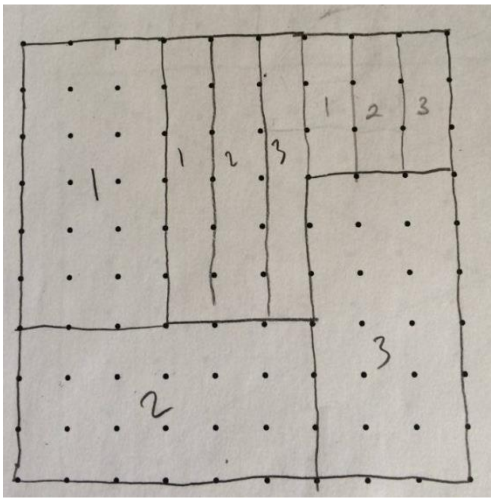


Figure 7. An example of a solution that was coded in Category 1

Category 2: Concrete Solutions

After PSTs were encouraged to cut the cake into three pieces only, the majority of the PSTs chose to divide the cake into thirds by counting the square units. Below are some of the solutions that were coded as concrete solutions.

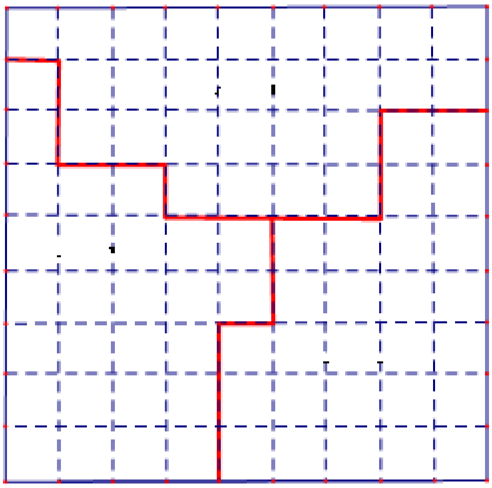


Figure 8. An example of a solution that was coded in Category 2

As can be seen in the above solution, the PSTs partitioned the cake into thirds with the area of 27 square units and the sides with frosting of 12 units, although the pieces were not congruent. They were able to see each piece as a compsite unit which consists of 27 singleton units. Similarly, another table came up with the solution below (see fig. 9). In both solutions that were coded as concrete solutions, PSTs justified their solutions by counting the units to make sure that the cake and the frosting were distributed evenly.

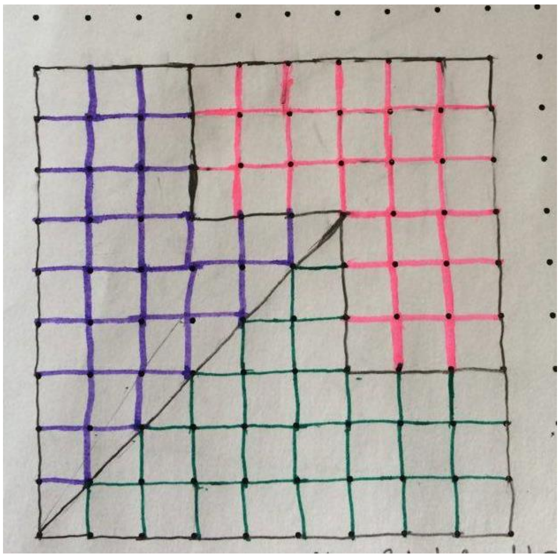


Figure 9. Another example of a solution that was coded in Category 2

Category 3: Abstract Solutions

Very few PSTs were able to construct a solution that did not require counting the provided units as it was the case in the above solutions. In the solution below, the PST argued that if the cake was cut into three pieces by making 120 degree angles from the center—where the two diagonals of the square intersect—, it would ensure that each person would recieve the equal amount of the cake and the frosting on the sides. Browning, Garza-Kling and Sundling (2008) provides one possible definition of an angle as "the amount of space that you turn between two rays that meet at a common vertex". They argue that defining angle simply as "two rays with a common endpoint" might prohibit students from further exploration of this abstract concept of angle. Although it was not stated explicitly, the PST who constructed the solution below argued that making congruent angles from a common vertex would ensure that the space between the lines would be equal.

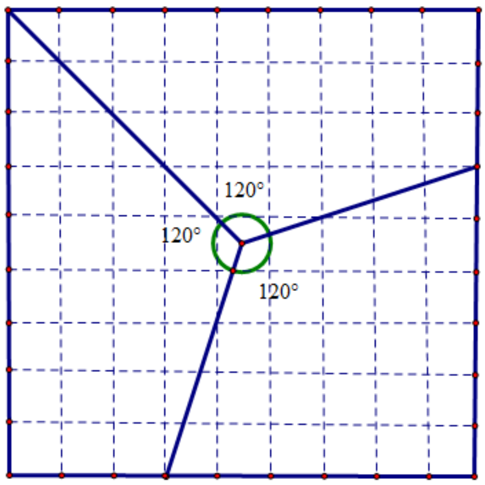


Figure 10. An example of a solution that was coded in Category 3

In another solution, the PST, who constructed the following solution (see fig. 11), was able to use her knowledge of area as a continuous attribute that can be divided into various discrete subunits. The PST argued that since all subunits—triangles—have the same area and each part has the same amount of subunits, the pieces are equal in size.

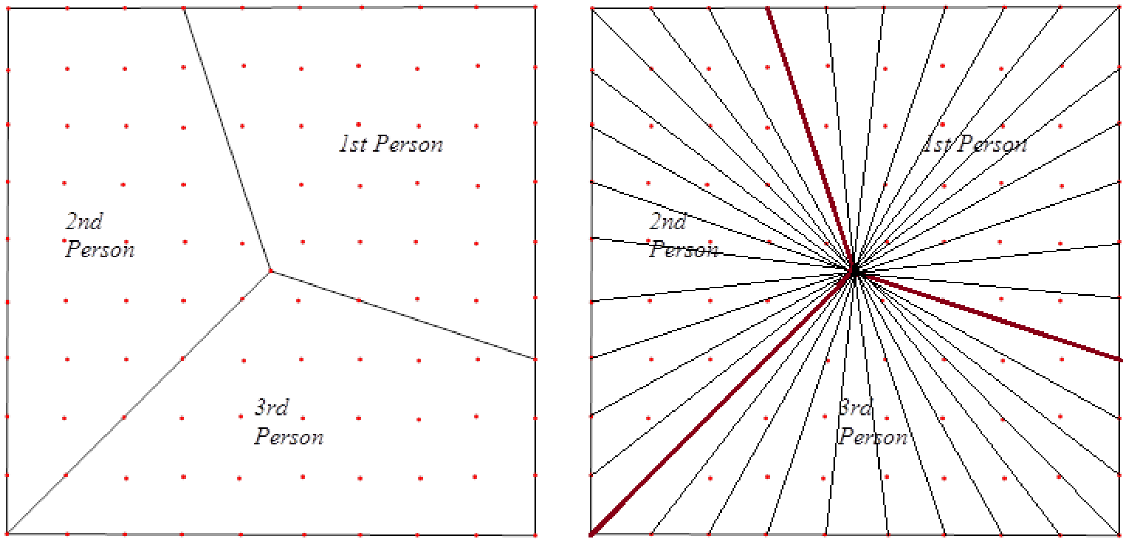


Figure 11. Another example of a solution that was coded in Category 3

Since these two solutions did not require simply counting the units of measure in order to justify the solutions, these two solutions were coded as abstract solutions.

Conclusion

In order for any struggle to be productive, these struggles with mathematics must be documented. However, the studies that aimed to document the nature of struggle is limited, especially in college classrooms. The aim of this study was to document what struggle looks like in a college classroom where pre-service mathematics teachers were engaged in solving a non-routine task. In this part the findings of the study will be summarized and discussed under the lights of existing literature regading to productive struggle.

Productive struggle is supported by a developmental progression in thinking and learning. This developmental progression can and should be nurtured in order for meaningful learning to occur in math classrooms. Students need support to learn how to move through a progression or range of solution methods as suggested by Fuson, Carroll and Drueck (2000). Hiebert and Grouws (2007) state that students' effort to make sense of mathematics, to figure something out that is not immediately apparent can advance their thinking and play an important role in deepening their understanding, if supported carefully toward a resolution and given appropriate time. This study documented the struggle types that PSTs encountered when they tried to figure out how to solve a challenging task in the classroom. The different kinds of struggles that PSTs encountered as they demonstrated effort to make sense of the task centered around when they encountered difficulty in figuring out how to get started or carry out their proposed solution to task, when they were unable to piece together and explain their ideas, or when they expressed an error or misconception as Warshauer described in her study (Warshauer, 2014b).Thus, the findings of this study aligned wellwith the resultsof Warshauer's study even if her study took part in a middle grade classroom. It would not be wrong to conclude then that struggle types are similar in all classrooms regardless of grade level or participants as long as participants are engaged in a high level task that has multiple solutions that are not appearent at first sight.

One of the struggle types documented during the class instruction was "express misconception". Given that PSTs, who participated in the group interviews, demonstrated such limited understanding of geometric concepts, especially knowledge about two-dimensional shapes and the inclusion relations of quadrilaterals, it was indeed not surprising to see these documented misconceptions came out during the class and served as a struggle type by inhibiting PSTs to proceed with the task. Although documenting pre-service teachers' misconceptions about the inclusion relations of quadrilaterals was not the primary focus of the study since this topic has already been investigated by many reserachers (see Fujita, 2012; Fujita & Jones, 2006; 2007; Zilkova, 2015), it was important to investigate possible misconceptions that the PSTs might hold in order to better describe the nature of this struggle type—express misconceptions.

In addition to four types of struggles, four types of instructor responses—Telling, Directed Guidance, Probing

Guidance, and Affordance—to the PSTs' struggles were also evident in this classroom episode.These responses of the instructor had different characteristics and roles during PSTs' productive struggle in the classepisode.Findingsof this study showed that the cognitive demand of the tasks was keptin the probing guidance and affordance types of responses since PSTs were afforded more time to work on the task or their thinking was probed during these responses. However, cognitive level of the task might be decreased to a lower levelin the responses of telling or the directed guidance since PSTs would be simply told what to do in telling or they would be guided towards a certain path. Zaslavsky (2005) argues that the process of seizing the opportunity to create uncertainty and doubt in learning mathematics is a powerful source, sothe potential of learning situations involving uncertainty should be at the core of classrooms. In order to create such uncertainity environment, learners should be allowed to experience this uncertainity and doubt during this conflict resolution as it is usually the case in the response of probing guidance and affordance. However, it does not mean that the responses of telling and directed guidance should be diminished from math classrooms. Instead, these responses should be mainly used to getting started or uncertainity in explaning types of struggles so that it might prevent lowering the cognitive level of the task while providing the needed support to the PSTs to continue to work on the task as it mainly happended during the class episode.

The selection and construction of worthwhile mathematical tasks is highlighted in the NCTM Professional Standards (1991) as one of the mostimportant pedagogical decisions a teacher needs to make. The tasks teacherspose in their classrooms is central to students learning as theyopen or close the students' opportunity for meaningful mathematics learning. Thus, mathematical tasks, the NCTM (1991) proposes, are not to be chosenlightly – because they are fun or suggested in textbooks. Instead, tasksshould be chosen because they have the potential to "engage students' intellect," can be approached in more than one interesting way," and "stimulatestudents to make connections and develop a coherent framework formathematical ideas" (NCTM, 1991, p. 25). This study demonstrated that implementing a high cognitive level task not only motivates PSTs to work on mathematics and persevere as solving the task, but it also provided opportunities for PSTs to connect important mathematical ideas to to construct as well as validate various solutions. Most recently with the implementation of the Common Core State Standards for Mathematics (CCSSM), expectations for what teachers should know go beyond merely facts and procedures, they must also understand the mathematical ideas embedded in procedures as well as apply procedures in different problem solving. In order for teachers to develop the core ideas as outlined in the CCSSM standards, they need support. This study showed that engaging PSTs in a high level doing mathematics task might provide this support to meet the expectationsas highlighted in the CCSSM.

Implications

This study aimed to describe the nature of struggle types that PSTs encountered when they were engaged in a high cognitive level task. In order for any struggle to be productive they should be described in details. Thus, one of the implications of this study is to provide guidance for teachers or college instructors to document learners' struggle types. In addition to the strguggle types, the responses of the instructor were also documented throughout this study. It is important to be aware of these responses types and their characteristics of supporting or impeding learning in a classroom setting. Thus, it is recommended for teachers and teacher educators to be attentive to learners' struggle types and to be aware of how to support learners to overcome these struggles without diminishing their learning opportunities.

Kapur (2010) has identified three conditions that promote a beneficial struggle as follows: (1) choosing problems to work on challenge without getting frustrated, (2) providing learners with opportunities to explain and elaborate on what they're doing and (3) giving learners the chance to compare and contrast good and bad solutions to the problems.In order to meet these conditionsmetioned by Kapur, a classroom culture that demonstrates struggle as a natural part of the learning process (Star, 2015), highlights the significant role of social interactions and exchanging personal viewpoints (Yackel & Cobb, 1996), and points perseveringas an important component of making sense of mathematics (CCSSM, 2010) should be established. Thus, teachers as well as teacher educators are encouraged to set classroom norms where struggling to make sense of mathematical ideas and persevering during this process are seen as natural and essential parts of the classrooms.

This study has documented that pre-service teachers hold limited understanding of essential geometric knowledge such as understanding the concepts of two-dimensional shapes and inclusive relations among them. This finding suggests at least two implications for educators. Firstly, knowing that students at all levels (college or below) struggle with geometric concepts especially inclusive relationships among quadrilterals, these concepts should be thought with such attention. Studies argue that teaching to identify quadrilaterals through an independent understanding prohibits functional understanding of the hierarchical classification of quadrilaterals (de Villiers, 1994; Lueng, 2008). De Villiers (1999) suggests that teachers should not allow students routinely to treat geometric concepts or properties to be memorized, nor as an aggregate of empirically discovered pieces of information. Similarly, Leung (2008) documents that defining sufficient and necessary conditions for being a quadrilateral with the aid of technology might be a helpful venue to teach this complex concept. In addition to thinking carefully regarding to how to teach this concept, as educators we should also consider about how to address students' limited knowledge or possible misconceptions that they might bring to our classrooms given that their misconceptions documented as one of the struggle types. Ball (1991) states that student errors can open a window into student understanding. Thus, as Ballargues teachers must move beyond right or wrong answers. Instead, they should see student errors as an opportunityto look into student thinking. Lampert (1992) and Rittenhouse (1998) have written about creating risk-free spaces in mathematics classrooms where students feel comfortable to make mistakes.

Students need sufficient time, not only to solve difficult mathematical problems, but also to develop genuine curiosity (Goldenberg, et. al., 2015). Thus, teachers need to carefully select tasks that require students not only to struggle, but also to develop curiosity. The selected tasks should not take learners through a narrow path to the one correct solution but to various solutions and provide the support that students need without diminishing the cognitive demand of the task. Challenging tasks that allowstudents tomakesense of the concepts, apply prior knowledge of formulas to new situations, and acquire conceptual knowledge through discourse and interaction, which are essential characteristics of active learning as highlighted in NCTM's Principles to Actions (NCTM, 2014) could promote learners' mathematical knowledge. In order to provide teachers and/or students opportunities to develop conceptual understanding of the concepts, these types of high cognive levels of tasks should be used in math classrooms.

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