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Procedural and Conceptual Difficulties with Slope: An Analysis of Students' Mistakes on Routine Tasks

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Abstract

This study extends past research on students' understanding of slope by analyzing college students' mistakes on routine tasks involving slope. We conduct both quantitative and qualitative analysis of students' mistakes on common slope tasks to extract information regarding procedural proficiencies and conceptual underpinnings required in order for students to reason successfully with various slope conceptualizations described in prior research. A case study analysis of the mistakes made by two students illustrates the importance of analyzing patterns of mistakes to reveal what conceptualizations of slope a student is fluent in working with. Results from this study delineate procedural proficiencies and conceptual underpinnings related to various slope conceptualizations that can help both teachers and researchers pinpoint students' understanding and make appropriate instructional decisions to help students advance their understanding.

Introduction

Functions play a crucial role throughout the mathematics curriculum. Students' earliest experiences with functions typically involve the study of linear relationships, building a foundation on which more advanced functional relationships are built (Nagle & Moore-Russo, 2014; NGA Center & CCSSO, 2010). The concept of slope is critical to the study of linear functions in beginning algebra and extends throughout the secondary mathematics curriculum to describe non-linear (e.g., quadratic and exponential) functions in advanced algebra (NGA Center & CCSSO, 2010; Yerushalmy, 1997), the line of best fit in statistics (Casey & Nagle, 2016), and the concept of a derivative in calculus (Stanton & Moore-Russo, 2012; Stroup, 2002).

In light of the important role that the concept of slope plays in students' understanding of both linear and non-linear functions, past research documenting students' challenges with slope are concerning. Studies have found both U.S. and international students have a minimal understanding of slope (Greens, Chang, & Ben-Chaim, 2007) and experience various conceptual difficulties (Hattikudur, Prather, Asquith, Knuth, Nathan & Alibali, 2011; Lobato & Siebert, 2002; Simon & Blume, 1994; Stump, 2001a; Stump, 2001b; Teuscher & Reys 2010; Zaslavsky, Sela, & Leron, 2002). Research has documented students' difficulties with interpreting slope in both functional and physical situations (Simon & Blume, 1994; Stump, 2001a) and with transferring knowledge of slope between problem types (Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Planinic, Milin-Sipus, Kati, Susac & Ivanjek, 2012). Additionally, research has shown that students struggle to make connections between slope and the notion of rate of change (Hattikudur et al., 2011; Stump, 2001b; Teuscher et al, 2010).

Perhaps one source of students' difficulties with the concept of slope is the variety of ways that it can be conceptualized. Moore-Russo and her colleagues (Moore-Russo, Conner, & Rugg, 2011; Mudaly & Moore-Russo, 2011) have refined and extended the conceptualizations of slope Stump (2001b) offered. Each of the resulting 11 conceptualizations (see Table 1, adapted from Moore-Russo et al. (2011)) has been documented among secondary or post-secondary students and instructors (Nagle & Moore-Russo, 2013; Nagle, Moore-Russo, Viglietti, & Martin, 2013).

Although conceptual understanding of slope is essential for understanding linear relationships, procedural knowledge of slope is also important. Students need a comprehensive knowledge of a procedure, along with an ability to make critical judgments about which procedure is appropriate for use in a particular situation (National Research Council, 2012; Star, 2005). According to Hiebert and Lefevre (1986), substantial understanding of mathematics includes connections between conceptual and procedural knowledge. Conceptual knowledge is knowledge that is rich in relationships, connecting new ideas to existing ideas, while procedural knowledge consists of formal language and symbol systems, as well as algorithms and rules (Stump, 2001a). Rittle-

Johnson, Siegler, and Alibali (2001) describe procedural knowledge as recognizing processes linked to particular problem types while conceptual knowledge is a more flexible understanding of governing rules that can be transferred to different problem types and representations.

Table 1. Concepts of slope

Category	Slope as…
Geometric ratio (G)	Rise over run of a graph of a line; ratio of vertical displacement over horizontal displacement of a line's graph
Algebraic ratio (A)	Change in *y* over change in *x*; ratio with algebraic expression, $\frac{y_2-y_1}{x_2-x_1}$
Physical property (P)	Property of line often described using expressions; steepness, slant, pitch, how high up, or it goes up
Functional property (F)	Constant rate of change between variables
Parametric coefficient (PC)	Coefficient *m* in equation $y = mx + b$
Trigonometric conception (T)	Tangent of a line's angle of inclination; direction component of a vector
Calculus conception (C)	Limit; derivative; tangent line to a curve at a point
Real-world situation (R)	Static, physical, dynamic, or functional situation (e.g., wheelchair ramp, distance versus time)
Determining property (D)	Property that determines whether lines are parallel, perpendicular or neither; property with which a line can be determined, if you are also given a point
Behavior indicator (B)	Real number with sign which indicates increasing (+), decreasing (-), horizontal (0) trends of line
Linear constant (L)	Constant property unique to straight figures

In the case of slope, procedural knowledge includes familiarity with the symbols typically used in relation to it (e.g., *m*), and the rules used to calculate it (e.g., $\frac{rise}{run}$ or $\frac{y_2-y_1}{x_2-x_1}$) (Nagle & Moore-Russo, 2013; Stump, 2001a). Conceptual knowledge of slope enables students to make connections between the various conceptualizations of slope and results in an ability to explain why particular procedures for calculating slope work. For instance, a student with a conceptual understanding of slope may visually apply a G*eometric Ratio* notion of slope as rise over run with a series of similar slope triangles to also conceptualize slope as a *Linear Constant* that is not dependent on the portion of the line at which one is looking (Nagle & Moore-Russo, 2013). This illustrates how conceptual knowledge and procedural skills are interconnected and form a web of reasoning (Egodawatte & Stoilescu, 2015; Kilpatrick, Swafford & Findell, 2001; Nesher, 1986; Rittle-Johnson & Koedinger, 2005). In a recent study of eleventh grade students' interconnected use of conceptual knowledge and procedural skills in algebra, Egodawatte and Stoilescu (2015) used error analysis to show how prevalent procedural errors sometimes indicated weak conceptual understanding. As described earlier, research has documented students' weak conceptual understanding of slope. However, findings that many students confuse *rise over run* and *run over rise* in the formula for slope and are unsure of the procedure to find a perpendicular line's slope also suggest that students may lack procedural knowledge of slope as well (Stump, 1999).

Table 2. Carlson and colleagues' levels of covariational reasoning

Covariation Level	Description of Mental Actions
L1: Coordination	Coordinating the *value* of one variable with changes in the other
L2: Direction	Coordinating the *direction of change* in one variable with changes in the other variable
L3: Quantitative Coordination	Coordinating the *amount of change* in one variable with changes in the other
L4: Average Rate	Coordinating the *average rate-of-change* of the function with *uniform increments* of change in the input variable
L5: Instantaneous Rate	Coordinating the *instantaneous rate-of-change* of the function with *continuous changes* in the independent variable for the entire domain

Since slope is the constant rate of change of two linearly related variables (*Functional Property* in Table 1), it is important to consider how students apply covariational reasoning as they conceptualize slope. Described as the "mental coordination of two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p. 354), covariational reasoning has been identified as a key prerequisite for advanced mathematical thinking (Carlson, Oehrtman, & Engelke, 2010; Confrey & Smith, 1995).

Carlson and colleagues (2002) describe five developmental stages of covariational reasoning, delineating how early experiences coordinating changes in two quantities builds to sophisticated interpretations of instantaneous rate of change. A summary of the reasoning levels and associated mental actions is provided in Table 2 (adapted from Tables 1 and 2 in Carlson et al., 2002).

The Present Study

Past research on slope has both described the multitude of ways which students might conceptualize it and described students' limited proficiency. However, these areas of research have not been merged. In particular, past research has not engaged in error analysis of students' solutions on common slope tasks to extract information regarding students' procedural and conceptual knowledge using the 11 slope conceptualizations. We conduct both quantitative and qualitative analysis of students' solutions to routine slope tasks in order to delineate procedural proficiencies and conceptual underpinnings that can be attributed to those mistakes. We then link these to the previously identified slope conceptualizations to provide greater insight into the procedural and conceptual knowledge that underlies each notion of slope. The research questions are:

1. What mistakes did students make when solving the various slope tasks?
2. Which tasks did students have the most trouble with and what mistakes were most prevalent?
3. What do students' mistakes reveal about their procedural proficiencies and conceptual understanding of slope?

Methods

In order to answer the research questions, we engaged algebra and precalculus students in solving routine slope tasks. Both qualitative and quantitative techniques were used to analyze students' responses to those tasks. A description of the participants, the assessment, and the data analysis follows.

Participants

Participants in this study were primarily college freshmen and sophomores at a single four-year college in the Northeastern region of the United States. Seven mathematics instructors representing 13 sections of Quantitative Reasoning (commonly known as Elementary Algebra), Algebraic Problem Solving (commonly known as College Algebra or Intermediate Algebra), and Precalculus agreed to administer the slope assessment to their students during class time. The instructors of these three courses were targeted since slope is a key topic in the first half of the curriculum of all three courses. The assessment was administered during the second half of the semester, after slope was taught. Students were given 30 minutes to complete the assessment and most students finished in the allotted time. Although the assessment was not graded, students received points for providing meaningful responses so most students were motivated to try their best on the problems. In all, 256 students completed the assessment with fairly even distribution among the three courses: Quantitative Reasoning (n = 79), Algebraic Problem Solving (n = 94), and Precalculus (n = 83). The 256 students represented 68.82% of all students enrolled in the 13 sections, with individual course participation percentages of 73.15% for Quantitative Reasoning, 75.2% for Algebraic Problem Solving, and 59.71% for Precalculus.

Assessment

The researchers developed a 15-question assessment containing standard slope questions similar to those that students solved on homework and exams. The tasks were purposefully familiar to students in an effort to reveal common procedural mistakes. Table 3 provides a summary of the assessment. The 15 questions belonged to six broad categories: (1) write an equation of a line given particular information, (2) write the equation of a line given its graph, (3) write the equation of a line given its graph and interpret in terms of a real problem situation, (4) use a table of values to write a linear equation, (5) determine whether graphs of two equations are parallel, perpendicular, or neither, and (6) sketch a line given particular information. The questions incorporate a variety of slope conceptualizations, as seen by the anticipated slope conceptualizations required to answer each question (Table 3). Note that since these were algebra and precalculus courses, the *Trigonometric* and *Calculus* conceptualizations of slope are not represented in this assessment.

Data Analysis

The 256 students each completed the 15-question assessment, yielding 3840 items to code. Coding began with one of the researchers grading all responses using a four-point scale: 4 points for a completely correct answer, 3 points for a mostly correct answer, 2 points for a half correct answer, 1 point for a partially (less than half) correct answer, and 0 points for a blank or nonsense answer. With 15 questions worth four points each, each student could earn 60 points overall.

Table 3. Slope assessment categories and question-specific details

Category Information	Specific Question Information	Anticipated Slope Conceptualizations
Find an equation of a line given information.	1. Given a point and a slope.	G, A, P, F, **PC**, R, D, B, L
	2. Given two points.	G, **A**, P, F, **PC**, R, D, B, L
	3. Given a point and a parallel line.	G, A, P, F, **PC**, R, **D**, B, L
	4. Given a point and a perpendicular line.	G, A, P, F, **PC**, R, **D**, B, L
Write the equation of a line given its graph.	5. Decreasing line on non-homogenous coordinate system.	**G**, A, P, F, **PC**, R, D, B, L
	6. Increasing line on non-homogenous coordinate system.	G, **A**, P, F, **PC**, R, D, B, L
Write the equation of a line given its graph *and* interpret it in the problem situation.	7. Decreasing line (time in months versus value of a TV).	G, **A**, **P**, **F**, **PC**, **R**, D, B, L
	8. Increasing line (number of units versus total cost to make).	G, **A**, **P**, **F**, **PC**, **R**, D, B, L
Use a table of values to write a linear equation.	9. Non-standard increments in *x* but *x*-values increase.	G, **A**, P, F, **PC**, R, D, B, **L**
	10. Non-standard increments in *x* and *x*-values alternate from increasing to decreasing.	G, **A**, P, F, **PC**, R, D, B, **L**
Determine whether graphs of two equations are parallel, perpendicular, or neither.	11. Standard form; lines are parallel.	G, A, P, F, **PC**, R, **D**, B, L
	12. Standard form; lines are perpendicular.	G, A, P, F, **PC**, R, **D**, B, L
Sketch a line given information.	13. Given a point and slope.	**G**, A, P, F, PC, R, D, **B**, L
	14. Given an equation in slope-intercept form.	**G**, A, P, F, **PC**, R, D, **B**, L
	15. Given an equation in standard form.	**G**, A, P, F, **PC**, R, D, **B**, L

After coding students' answers on the four-point correctness scale, the researchers used grounded theory (Glaser & Strauss, 1967) to code students' solutions for mistakes. For every answer that did not receive a perfect score, the researchers analyzed the students' solution to determine what mistake(s) were made. We define a mistake as a wrong action or inaccuracy or lack of action that was demonstrated in the problem solution. We recognize that the same mistake may stem from different sources of misunderstanding and we do not distinguish between these when coding for mistakes. Based on the students' solutions, we generated a list of possible mistakes. When a new solution suggested the need for an additional mistake coding, the code was added to the list and all responses were revisited in light of the revised list. After working together to generate a list of possible codes using grounded theory, one researcher revisited all student work and completed the coding according to the list of mistakes. After rating and analyzing responses of all participants, all results were input to Microsoft Excel and analyzed with proper statistical methods.

The final phase of data analysis involved an in-depth analysis of two students' mistakes across the slope assessment. These two students were selected because they made similar mistakes across the slope assessment and had similar overall scores on the assessment, yet they had different patterns in the frequency of these mistakes. At a surface level, one might see these students' scores and common mistakes and conclude they have similar levels of understanding of slope. For this part of the analysis, the researchers revisited the mistakes on each problem and studied them collectively to reveal what underlying understandings and misconceptions about slope the students' responses revealed. In order to do this, the researchers created a table outlining the mistakes made on each problem with an interpretation of the mistakes in the context of the problem. Then, the researchers read across questions to form a comprehensive picture of what knowledge and misunderstandings the students demonstrated across the slope assessment.

Results

Classifying Mistakes

In order to answer the first research question, we conducted a thorough analysis of the mistakes that students made when solving the slope tasks. In total, 18 mistakes emerged from the grounded theory approach to coding students' solutions on the slope tasks. Table 4 provides a list description of all such mistakes and indicates the assessment question(s) on which the mistake was made as well as the frequency of the mistake across all students and questions.

Table 4. Mistake Codes, Related Questions, and Frequency

Code #	Abbreviation code	Description of Mistake	Related Questions	Frequency
1	*NoResponse*	No response or nonsense answer	All questions	496
2	*Arithmetic*	Any type of addition, subtraction, multiplication, or division mistake	All except 14	310
3	*SimpleFraction*	Not changing a fraction to the simplest form	All except 1, 3, 13, 14	128
4	*NoXvariable*	Don't put the *x* variable after the slope in the equation	All except 6, 11, 12, 13, 14	54
5	*SlopeRunRise*	Calculating a slope as run/rise instead of rise/run	2, 5, 6, 7, 8, 9, 10	57
6	*CoordiPoints*	Calculating $\frac{y_2-y_1}{x_1-x_2}$, hence getting the opposite of the actual slope.	2, 5, 6, 7, 8, 9, 10	17
7	*SubtractCoord*	Calculating $\frac{y_2-x_2}{y_1-x_1}$	2, 8, 9, 10	8
8	*OppSignSlope*	Putting a negative sign for an increasing line's slope or vice versa	5, 6, 7, 8	95
9	*BlockSlope*	Using blocks instead of axis' units to calculate a slope	5, 6	94
10	*MentalAction1*	Does not coordinate the *value* of one variable with changes in the other variable	7, 8	32
11	*MentalAction2*	Does not coordinate the *direction* of change in one variable with changes in the other variable	7, 8	30
12	*MentalAction3*	Does not coordinate the *amount* of change in one variable with changes in the other variable	7, 8	118
13	*CalcYintercept*	Don't know how to calculate the *y*-intercept with many non-routine points	9, 10	101
14	*NoSlopeInter*	Not revising a standard form to a slope-intercept form when using the coefficient of *x* as the slope	11, 12	55
15	*GraphOpposite*	Graphing opposite direction with a given slope	13, 14, 15	73
16	*PlotXYchange*	Plotting a point using *x*-coordinate value as a *y*-coordinate and vice versa	13, 14, 15	29
17	*NoOppPerp*	Using reciprocal but not opposite slope to apply to the perpendicular line's slope	4	32
18	*NoRecPerp*	Using same slope to apply to the perpendicular line's slope or just put opposite sign	4	29

A brief description of the coding guidelines for each mistake follows. Please refer to Figure 1 for sample responses illustrating each of the 18 mistakes.

NoResponse was associated with all 15 questions and indicates that a student either did not attempt the problem or provided a completely nonsense response which showed no meaningful interpretation of the problem at hand (Sample Response 1). *Arithmetic* appeared on all questions except one (#14) and indicated that a student made a mistake related to an arithmetic calculation (Sample Responses 2, 4, 5). This code often appeared during multiplication and division when students had sign errors (e.g., multiplying two negative numbers yields a negative result), and during addition or subtraction when basic calculation or sign errors were made.

SimpleFraction was associated with the majority of the questions (all except 1, 3, 13, 14) and indicated that a fraction was not written in simplified form (Sample Responses 2, 4). It is important to mention that the researchers decided to code this as a 'mistake' even though students received full-credit for a response that was correct but not in simplified form. Because it was seen very frequently while coding responses, the researchers chose to keep it as a cateogry and explore whether it was connected to other mistakes or showed other interesting trends. *NoXvariable* was seen on the majority of questions (all except 6, 11, 12, 13, 14) and indicates that a student did not write the variable x when providing a linear equation in slope-intercept form (i.e., $y = m + b$ rather than $y = mx + b$, Sample Response 3).

Three mistakes related to students' calculations using the change in y over change in x or rise over run slope ratio. *SlopeRunRise* related to questions 2, 5, 6, 7, 8, 9, and 10. This mistake indicates that students used the reciprocal ratio when calculating slope, using $\frac{run}{rise}$ or $\frac{x_2-x_1}{y_2-y_1}$ instead of their correct counterparts (Sample Response 4). A similar mistake, *CoordiPoints*, related to the same questions. This was seen when students failed to correctly coordinate x and y-coordinates, calculating $\frac{y_2-y_1}{x_1-x_2}$ or $\frac{rise}{-run}$ and hence finding the opposite of the actual slope (Sample Response 5). *SubtractCoord* related to questions 2, 8, 9, and 10. This mistake was coded when a student subtracts corresponding x and y-coordinates, that is $y_2 - x_2$ instead of $y_2 - y_1$ or $y_1 - x_1$ instead of $x_2 - x_1$ (Sample Response 2).

OppSignSlope related to questions 5, 6, 7 and 8, which all asked for the equation of a line provided graphically. Questions 5 and 7 portrayed decreasing lines with negative slope while questions 6 and 8 portrayed increasing lines with positive slope. This mistake was coded when students provided a slope with the wrong sign for the given line (i.e., positive slope on questions 5 and 7 and negative slope on questions 6 and 8, Sample Response 5).

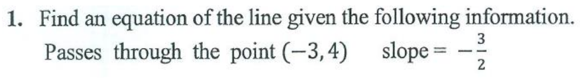
BlockSlope related to questions 5 and 6 where the graphs had non-homogenous coordinate systems. For example, on question 5, the unit of one block of the y-axis was 4 and the unit of one block of the x-axis was 2. Since the graph went down 5 blocks and over 3 blocks, some students calculated the slope as $-\frac{5}{3}$, not taking into account the units to yield the correct slope of $-\frac{20}{6} = -\frac{10}{3}$ (Sample Response 3).

MentalAction1, *MentalAction2*, and *MentalAction3* related to questions 7 and 8, which asked students to explain what the equation meant in terms of the real-world situation provided. This coding is based on Carlson and colleagues' (2002) covariational reasoning framework (recall Table 2). In particular, interpreting slope in terms of the real-world variables requires reasoning about (a) the covariation of the two variables being considered (L1 Covariational Reasoning), (b) the direction of change of the two quantities (L2 Covariational Reasoning), and (c) the amount of change in one quantity per set change in the other quantity (L3 Covariational Reasoning). A code of *MentalAction1* indicated that a student did not demonstrate knowledge of the two covarying quantities. This was often seen in responses that considered only a single variable or just read off the equation with variables replaced with real-world quantities (Sample Response 6). A code of *MentalAction2* indicates that a student *did* demonstrate L1 covariational reasoning but either did not attempt or made errors in L2 covariational reasoning. This generally appeared when students described the direction of change incorrectly (e.g., "the value of the HDTV increases as the number of month increases"). The *MentalAction3* code indicates that a student demonstrated both L1 and L2 covariational reasoning but either did not attempt or made an error when reasoning using L3 covariational reasoning. Generally, this code indicated that a student did not attend to the amount of change (e.g., "the value of the HDTV decreases over time") or did not correctly interpret the slope as a ratio of change in y variable over unit change in x variable (e.g., "For every 15 units the cost to make toys increases by 400", Sample Response 8). Because covariational reasoning has been noted as a key prerequisite notion for much of mathematics, we conducted additional analysis of how students' covariational reasoning levels were related to their overall performance on the slope tasks. This is provided in the following section.

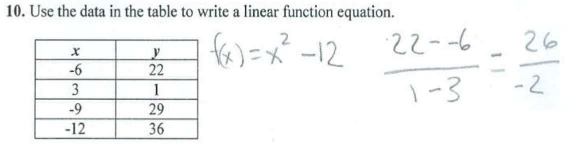
CalcYintercept related to questions 9 and 10 on which students were asked to use a table of values to find the linear equation. Many students used two points to determine the slope but then did not attempt to find the y-intercept or made errors when doing so, resulting in this code (Sample Response 2). *NoSlopeInter* related to questions 11 and 12, which asked students to determine whether the graphs of two linear equations were parallel, perpendicular or neither given two equations in standard form. This code indicates that a student did not convert the standard form to slope-intercept form before taking the coefficient of x as the slope (Sample Response 9).

GraphOpposite related to questions 13, 14, and 15 in which students were given certain information and were asked to graph the corresponding line. This code indicates that the students oriented the line in the wrong direction, either graphing an increasing line when the line should have been decreasing (based on having a negative slope) or graphing a decreasing line when the line should have been increasing (based on having a positive slope, Sample
Response 10). Notice that this code may be closely related to *OppSignSlope*, which indicates that students made a similar error working from the graphical to analytic representations. *PlotXYchange* also related to questions 13,

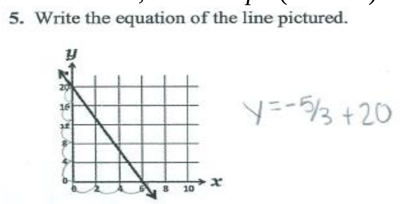
Sample Response 1
NoResponse (Score 0)



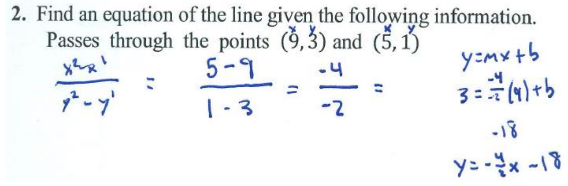
Sample Response 2
Arithmetic, *SimpleFraction*, *SubtractCoordi*, *CalcYintercept* (Score 0)



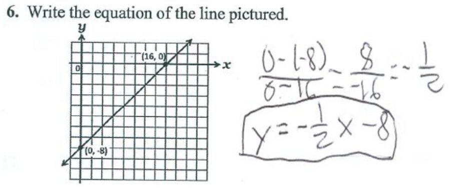
Sample Response 3
NoXvariable, *BlockSlope* (Score 1)



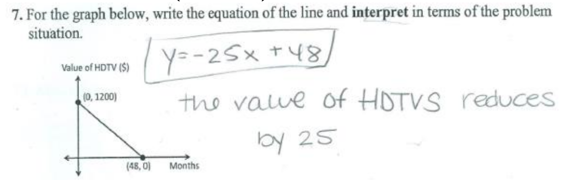
Sample Response 4
Arithmetic, *SimpleFraction*, *SlopeRunRise* (Score 1)



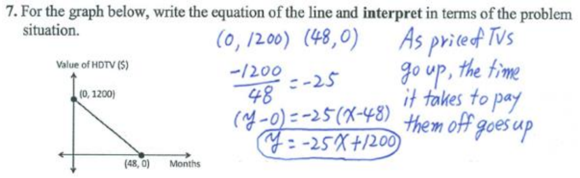
Sample Response 5
Arithmetic, *CoordiPoints*, *OppSignSlope* (Score 2)



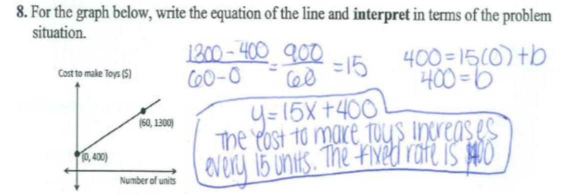
Sample Response 6
MentalAction1 (Score 2)



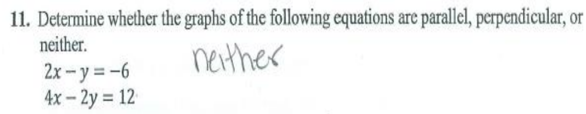
Sample Response 7
MentalAction2 (Score 2)



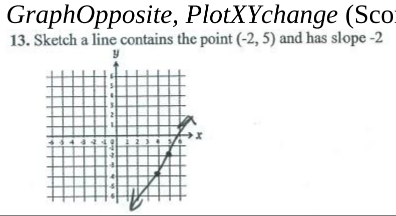
Sample Response 8
MentalAction3 (Score 3)



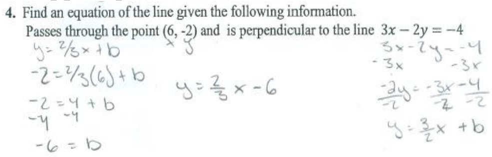
Sample Response 9
NoSlopeInter (Score 0)



Sample Response 10
GraphOpposite, *PlotXYchange* (Score 0)



Sample Response 11
NoOppPerp (Score 1)



Sample Response 12
NoRecPerp (Score 1)

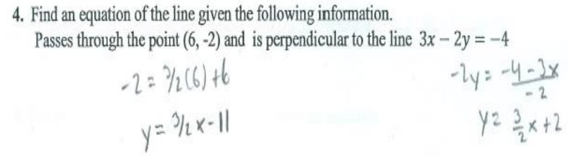


Figure 1. Sample responses of coding and scoring (all scores out of 4 possible points)

14, and 15 where students were graphing lines. This code indicated that a student interchanged the *x* and *y*-coordinates when graphing (Sample Response 10). For instance, rather than plotting the point (-2, 5), a student may have interchanged the horizontal and vertical axes, resulting in plotting of the point (5, -2).

The last two codes related only to question 4, which asked students to find a line through a given point and perpendicular to a given line. *NoOppPerp* indicates that a student used just the reciprocal of the slope of the given line, not its negative reciprocal (Sample Response 11). *NoRecPerp* refers to a response that gave either the same slope or the opposite slope of the perpendicular line, without ever taking the reciprocal of the slope (Sample Response 12).

Covariational Reasoning and Overall Performance

Recall that Questions 7 and 8 provided an opportunity to observe the level of covariational reasoning students used when interpreting the meaning of slope in light of the relationship between the real world variables. We conducted additional analysis to see how students' level of covariational reasoning correlated with their overall score on the slope assessment. As expected, students who exhibited higher levels of covariational reasoning scored higher on the slope assessment as a whole. First, demonstrating fluency with L3 covariational reasoning on both Question 7 and Question 8 was correlated with a higher overall score on the slope assessment (r = 0.294). Interestingly, occurrence of *MentalAction3* was also positively correlated with overall score (r = 0.203). Recall that *MentalAction3* was coded when an individual demonstrated L2 reasoning but failed to demonstrate L3 reasoning. While not as highly correlated with average score as L3 reasoning, the results suggest that the development of L2 reasoning is an important predictor of student success with routine slope tasks. On the flip side, occurrence of *MentalAction1* and *2* were negatively correlated with average percentage score (r = $-0.005, -0.03$, respectively). Again, these suggest that students should develop at least L2 covariational reasoning in order to engage successfully with slope tasks.

Overall Performance on Slope Problems

In order to answer the second research question, we summarized overall performance on slope problems. As mentioned previously, each student received a score out of 60 possible points on the 15-question slope assessment. The mean percentage score for all 256 students was 65.66%, with Algebraic Problem Solving students scoring highest (66.76%), Precalculus students scoring in the middle (65.13%), and Quantitative Reasoning students scoring lowest (64.92%). A single factor ANOVA showed no significant difference on overall percentage based on the students' course of enrollment [F (2, 253) = 0.15, p = 0.86 >> 0.05]. It is interesting that not only did Precalculus students not score significantly higher than students in the more basic Algebraic Problem Solving and Quantitative Reasoning courses, but they actually scored lower in overall percentage (albeit not statistically significant) compared with the Algebraic Problem Solving students. One possible explanation for this result is that less time is spent on teaching slope in the Precalculus course since it is assumed students have worked with the concept in algebra courses in the past.

Four Questions Which Have Lowest Average Percentage Scores

Among 15 questions, the four lowest average percentage scores were Questions 10 (45.4%), 4 (54%), 7 (55%), and 8 (55.5%). Figure 2 illustrates sample responses highlighting typical mistakes for these four questions. Despite being a very standard task, students scored the lowest on Question 10. Many students made a mistake when coordinating points in the slope formula (see response 10a), resulting in a positive slope instead of a negative slope. Interestingly, this mistake was not as common on Question 9, which also provided students with a table of values and asked for the linear equation. This discrepancy suggests that perhaps the non-monotonic *x*-coordinates in the table on Question 10 caused some of the difficulties for students. Another common mistake on this task is illustrated in solution 10b. Although this student coordinated *x*- and *y*- coordinates appropriately, an arithmetic error occurred where the student failed to write $-(-9)$ in the denominator of the slope ratio.

Question 4 was the second most difficult problem, with common mistakes illustrated in Figure 2. Response 4a illustrates the common mistake of calculating the *y*-intercept before finding the perpendicular line's slope. Although this solution uses the negative reciprocal slope of -2/3 in the final slope-intercept form of the equation, notice that the original slope of 3/2 was used when calculating the slope-intercept of the perpendicular line. The variable *x* is also omitted from the slope-intercept form of the equation. The mistake illustrated in response 4b was also common; the equation was changed to the slope-intercept form correctly but the slope was only negated (no reciprocal was taken) to find the slope of a perpendicular line.

Questions 7 and 8 both required students to write an equation (given a graph) and interpret the equation in light of the real world context that was provided. These problems also proved difficult for students, with similar mistakes made on both problems.

Response 7a was typical for these problems since many students correctly calculated the equation but were unable to apply L3 covariational reasoning to describe the slope in terms of the relationship between months and the value of an HDTV. Response 7b contains a sign error when calculating slope, hence the equation for the line was also incorrect. Interestingly, this student's interpretation that "…every month the value of HDTV decreases by $25" is correct and suggests that the student relied on the graph to get the inverse relationship. Regarding question 8, the equation in Response 8a was correct, but the interpretation "15 is the # of units, 400 is the amount per toy" was not. In Response 8b, the student used the incorrect slope formula $\frac{x_2-x_1}{y_2-y_1}$, so the result was $\frac{1}{15}$ instead of 15. In addition, the interpretation of the equation demonstrated L2 reasoning but inaccurately described the amount of change (resulting in the *MentalAction3* code).

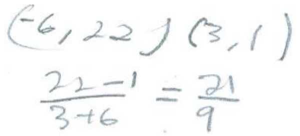
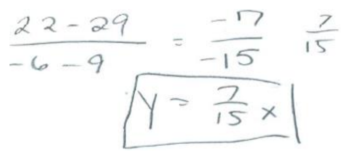
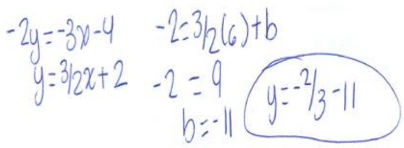
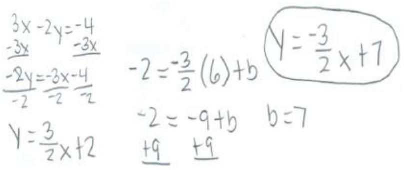
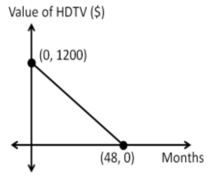
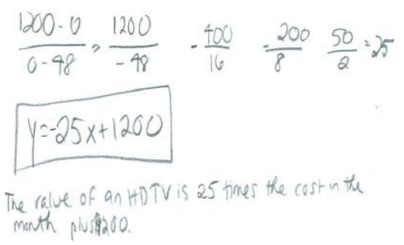
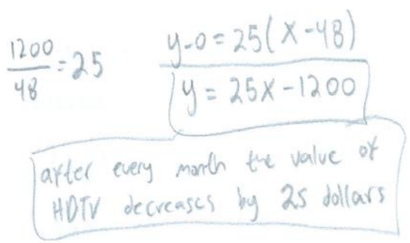
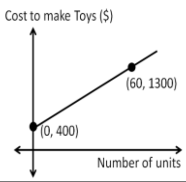
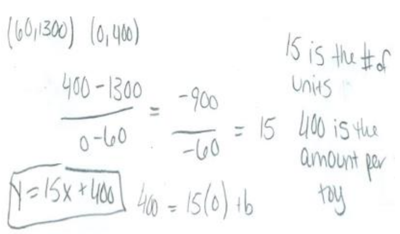
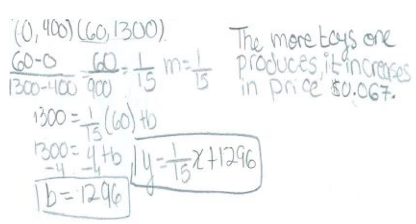
10. Use the data in the table to write a linear function equation. **x** / **y**: -6, 22; 3, 1; -9, 29; -12, 36	10a. *CoordiPoints, SimpleFraction* 	10b. *Arithmetic* 
4. Find an equation of the line given the following information. Passes through the point (6, -2) and is perpendicular to the line $3x - 2y = -4$	4a. *NoRecPerp, NoXvariable* 	4b. *NoRecPerp* 
7. For the graph below, write the equation of the line and interpret in terms of the problem situation. 	7a. *MentalAction1* 	7b. *CoordiPoints* 
8. For the graph below, write the equation of the line and interpret in terms of the problem situation. 	8a. *MentalAction1* 	8b. *SlopeRunRise, MentalAction3* 

Figure 2. Sample responses for questions on which students received the lowest average score

What students' mistakes reveal about their understanding of slope

While the above coding of mistakes reveals only what inaccuracies we found in students' answers, carefully examining the pattern of student mistakes over the course of the slope assessment can reveal important information about a student's conceptions of slope. In particular, a simple isolated incident may mean a student made a procedural slip while repetition of a mistake across problem types and representations may indicate deep-rooted conceptual misunderstandings (Egodawatte & Stoilescu, 2015). We provide a case study analysis of the mistakes made by two students on the slope assessment to illustrate. See Figure 3 and 4.

Stephanie (pseudonym) was a Quantitative Reasoning student and her overall score was 29 out of 60, while Carly (pseudonym) was in the Algebraic Problem Solving course and her score was 31 out of 60. Despite their similar scores on the slope questions, a detailed analysis of their responses show they had different understandings of slope.

Case Analysis of Stephanie

Stephanie's solutions on questions 1, 3, 11, and 12 indicate that she can algebraically manipulate an equation into slope-intercept form, use the *Constant Parameter* notion of slope to plug *m* into the slope-intercept form of an equation, and calculate the *y*-intercept of a linear equation. Although Stephanie uses just a reciprocal slope (not a negative reciprocal) when finding the perpendicular line on question 4, her correct response to question 12 suggests that she might have made a simple procedural error on question 4.

By contrast, Stephanie shows a pattern of mistakes on questions 2, 5, 6, 7, 8, and 9 that suggest she has a conceptual misunderstanding of slope as the ratio of the run over rise, rather than the rise over run. This is particularly concerning because Stephanie used the incorrect ratio on problems which used analytic (questions 2, 9) and graphical representations (questions 5, 6, 7, 8). The repetition of this mistake across representations supports the conjecture that this is a deep-rooted misunderstanding related to the *Algebraic Ratio* and *Geometric Ratio* conceptualizations of slope.

Also, despite Stephanie's procedural fluency with algebraic manipulation and calculating *y*-intercepts, her responses to questions 5, 7, and 8 suggest that she does not conceptually understand how *b* in the equation is related to the *y*-intercept of the graph. In particular, for each of these questions, Stephanie's incorrect slope formula resulted in an equation with the wrong *y*-intercept. While we cannot say for sure that Stephanie did not experience any cognitive conflict when comparing her equation to the graph provided, she did not appear to change her answer or suggest that her answer did not seem right based on the graphs. As a result, it seems that Stephanie may be procedurally proficient with calculating the *y*-intercept but not necessarily have conceptual grounding in its meaning.

Similarly, on questions 13 and 14, Stephanie sketched an increasing line instead of a decreasing line for an equation with a negative slope. Since Stephanie correctly sketched the increasing line on question 15, it appears that she was not considering the sign of slope as it relates to a line's orientation on the *xy*-plane. This suggests she does not have a strong conceptual understanding of slope as a *Behavior Indicator*.

In summary, Stephanie is algebraically adept at manipulating equations and thus has success on tasks which require manipulation to identify *m* or find *b*. Her ability to point to *m* in a slope-intercept form of an equation shows a procedural proficiency with the *Constant Parameter* conceptualization of slope. However, the consistency of mistakes across other problem types suggest her understanding of slope is limited to this very procedural interpretation of slope as a *Constant Parameter* with a lack of understanding about what that parameter indicates about the rate of change of two covarying variables or their graphical representation.

Case Analysis of Carly

By contrast with Stephanie, Carly's responses across the slope questions suggest she has a solid grasp with the *Geometric Ratio* and *Algebraic Ratio* conceptualizations of slope. We draw this conclusion despite a few mistakes related to the ration notion on slope. In particular, Carly uses the ratio of run over rise on Questions 6 and 8 and appears to misread the graph on Question 7 (using a rise of 24 instead of a rise of 20 for a run of 6). Despite these mistakes, Carly correctly applies both the *Geometric* and *Algebraic Ratio* conceptualization of slope throughout the other tasks suggesting she may have simply made a procedural error on the tasks mentioned (perhaps even calculating in her head as suggested by her lack of work to support her answer). In particular, she demonstrates an understanding of the *Algebraic Ratio* conceptualization of slope on her responses to Questions 2, 9, and 10 and *Geometric Ratio* on Questions 5(despite procedural error), 7, 13, 14, and 15.

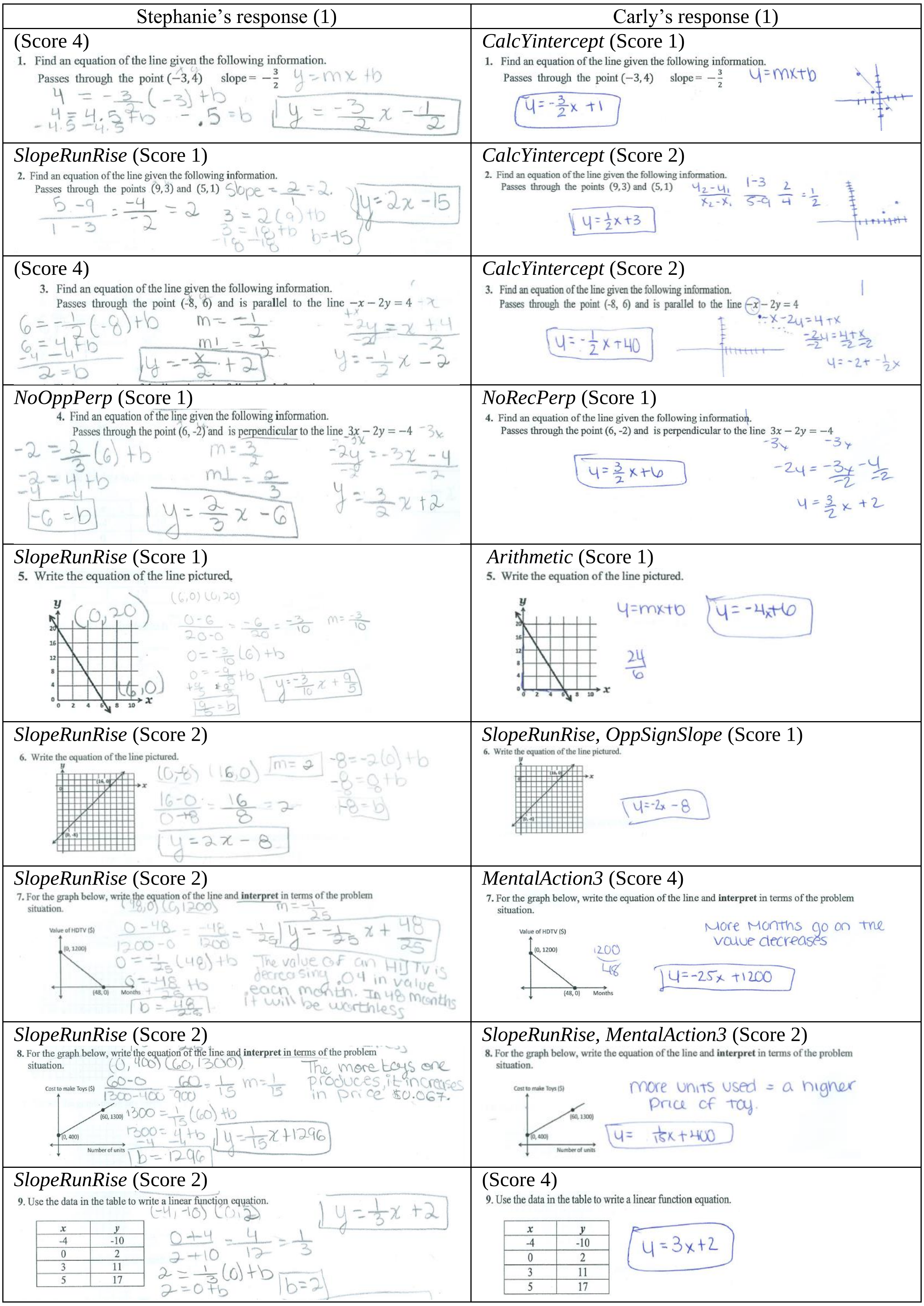


Figure 3. Analysis of the mistakes made by two students

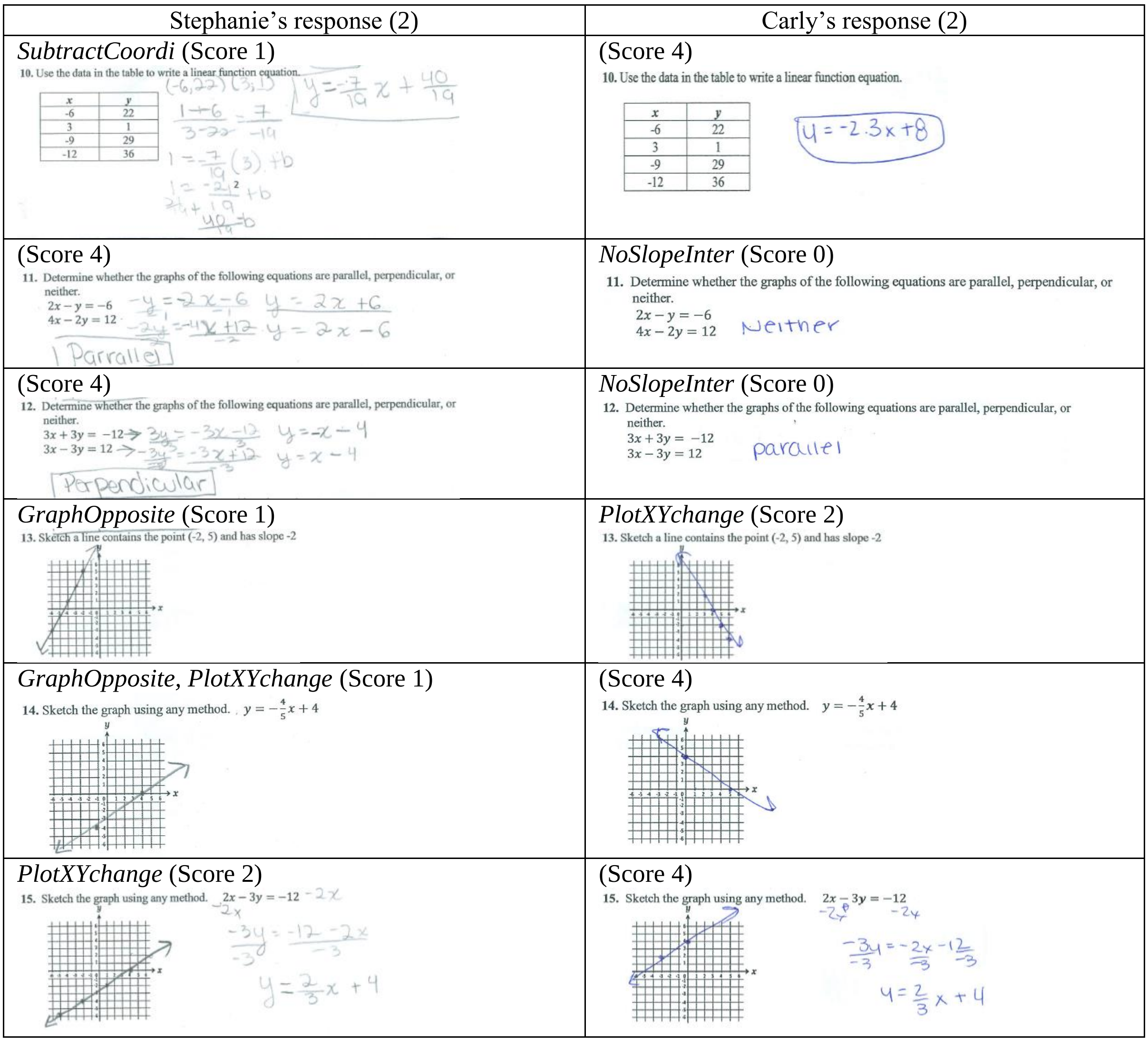


Figure 4. Analysis of the mistakes made by two students

Carly's responses also seem to highlight a strong visual emphasis when interpreting slope. Careful analysis of her responses suggest that she tended to reason visually when interpreting the slope and *y*-intercept of an equation. For instance, it appears that perhaps Carly's incorrect answer on Question 1 stems from her reliance on the graphical representation of her line to find the *y*-intercept. While this approach is acceptable, Carly's graph lacked precision and therefore her *y*-intercept was inaccurate. Despite this mistake, her approach shows a strong connection between the visual representation and the analytic representation via the slope-intercept form of the equation. This is in stark contrast to Stephanie who we never saw making explicit connections between the slope in different representations. Carly's graphical approach can also be seen on Questions 14 and 15. On Question 14, Carly graphs the *y*-intercept at (0,4) and appears to use the slope to find a second point by moving down four units and right 5 units from the *y*-intercept. Interestingly, Carly uses a similar approach to graph the line on Question 15 but appears to interpret the slope of 2/3 as warranting a move down two units and left three units from the *y*-intercept of 4. Her flexibility with interpreting the slope ratio of 2/3 as a -2/-3 further supports her strong *ratio* conception of slope.

Despite her strong grounding of slope as a ratio that cuts across representations, recall that Carly still did not score well overall. Looking for trends in her mistakes, Carly has a series of inconsistencies that suggest she may have made several procedural mistakes on the tasks. On two occasions (Questions 2, 5), Carly used the *x*-intercept instead of the *y*-intercept as *b* in the slope-intercept equation of the line. In both instances, it appears that Carly used the graph of the line to determine the *y*-intercept but used the coordinate where the graph crossed the *x*-axis instead of the *y*-axis. Given her correct interpretation of the *y*-intercept on other tasks (Questions 8, 14 and 15), we conjecture her mistakes on Questions 2 and 5 may have been careless mistakes rather than indicative of misunderstanding the graphical interpretation of the *y*-intercept. An interesting contrast to her overall success finding *y*-intercepts graphically, Carly struggled to find the *y*-intercept on Questions 3 and 4. In particular, on these tasks Carly did not have enough information to produce a graphical representation with two points indicated. It seems that while Carly was generally comfortable interpreting a *y*-intercept visually, she struggled to find one analytically. This suggests a possible disconnect between the visual and analytic representations of *y*-intercepts and supports our previous claim that Carly tends to think visually.

Carly's incorrect responses on Questions 11 and 12 may indicate that she struggles to interpret slope as a *Determining Property*. However, based on her responses, it seems likely that Carly correctly interprets equal slopes as indicating lines are parallel (also supported by her use of equal slopes in Question 3) but fails to realize she first needs to write the equation in slope-intercept form. Thus, we interpret this more as a weakness in a *Parametric Coefficient* interpretation of slope than in *Determining Property*. Overall, Carly demonstrates a strong tendency to use visual approaches to slope while conceptualizing slope as both a *Geometric* and *Algebraic Ratio*. Carly makes several procedural mistakes and careless errors throughout the tasks and struggles with the algebraic manipulation. Despite a strong grounding in slope as a *ratio*, algebraic manipulation and procedures hindered Carly's ability to respond to the slope tasks.

Discussion

Our study of students' mistakes on routine slope tasks has built on previous literature related to slope by analyzing particular mistakes that may hinder students' abilities to reason successfully with the various slope conceptualizations. Over the 3,840 responses we coded, many mistakes were made. A total of 18 mistake categories emerged from the grounded theory approach to coding students' solutions. The variety of mistakes we observed indicates that there are many procedural proficiencies and conceptual underpinnings required for students to work successfully with the various slope conceptualizations. Arithmetic mistakes, including addition, subtraction, multiplication and division operations were the most widespread mistakes (other than blank responses) regardless of a student's class of enrollment. These arithmetic errors carried over into algebraic manipulation with many students making mistakes when adding or subtracting a variable term to the other side of the equation or dividing by the coefficient of the *x*-term when converting from standard to slope-intercept form. This is an important reminder that even when a student has a strong conceptual grasp of slope, a lack of procedural proficiency with basic underlying ideas may hinder his or her ability to reason successfully on slope tasks.

Interpreting Students' Mistakes

Our research has also revealed the importance of analyzing students' mistakes and looking for trends when making judgements about a student's understanding of a particular mathematical topic. This was illustrated through the case study analyses of Stephanie and Carly. Although the students score similarly on the slope assessment and even made some of the same mistakes on particular questions, analysis of the mistakes made across the tasks revealed very different understanding of slope. In particular, Stephanie's responses showed she lacked any conceptual understanding of slope as a ratio and her correct answers stemmed primarily from her ability to manipulate algebraic equations and to recognize slope as m in an equation of the form $y = mx + b$. Thus, Stephanie had a procedural understanding of slope as a *Parametric Coefficient* and procedural proficiency with manipulating algebraic equations. However, her pattern of mistakes across the questions revealed she held a misconception of slope as run over rise and suggested she could not move between different representations of slope. Carly, on the other hand, made several procedural errors throughout the tasks but careful analysis revealed that she held strong *Algebraic* and *Geometric Ratio* conceptualizations of slope and she was quite adept at moving between representations although she seems to prefer visual interpretations. This is critical information for both teachers and researchers as we seek to better understand students' current concept attainment in order to provide appropriate instruction to move that understanding forward. Since students' mistakes are the visible cues to their understanding, it is important to recognize what information about students' underlying understanding can be revealed from their mistakes over a series of tasks.

Slope Questions for Instruction

The questions on which students had the most difficulty can also provide important insight to teachers as they design tasks for their own classrooms. Results suggest that teachers should consider including tables with *x*-values that have varying increments and which are non-monotonic. This is supported by students difficult with

Question 10, a seemingly standard question other than the lack of a pattern in the *x*-coordinates provided in the table. Students' difficulties with Questions 7 and 8 highlight the need for teachers to link the *Algebraic* and *Geometric Ratio* conceptualizations with the *Functional Property* idea of slope as a rate of change of two covarying quantities. Many students struggled on these examples because although they were able to explain that the two variables changed together, many even describing the corresponding directions of change in the variables, they struggled to interpret the slope as the amount of change in the dependent variable per a unit change in the independent variable. Thus, our results remind teachers that L3 covariational reasoning is a conceptual underpinning that helps to link the *Functional Property* conception of slope as the rate of change of two variables with *Behavior Indicator* and *Physical Property* conceptions of slope that focus on the direction and magnitude of change, respectively.

Procedural Proficiencies and Conceptual Underpinnings of Slope Conceptualizations

Past research has focused on describing the many different conceptions of slope. Our study has elaborated on past research by beginning to describe the vast procedural knowledge and conceptual underpinnings that are associated with the many ways to conceptualize slope. For instance, the case analyses of Stephanie and Carly suggest that students may struggle on slope tasks either because of a lack of conceptual understanding of slope as one or more of the basic slope categories, or as a result of procedural mistakes required to carry out processes associated with the slope conceptions. As a result of students' mistakes on the various slope questions, we have developed a preliminary list of procedural proficiencies and conceptual underpinnings related to each of the slope conceptualization categories. This is an important step which allows teachers and researchers to begin to break down the underlying ideas and practices that are necessary for a student to work fluidly with a particular notion of slope. A preliminary list of the underlying procedural proficiencies and conceptual underpinnings associated with each category of slope reasoning is provided in Table 5.

Table 5. Procedural Proficiencies and Conceptual Underpinnings for Each Category

Category	Procedural Proficiencies	Conceptual Underpinnings
Geometric ratio (G)	Count "units" for vertical change. Count "units" for horizontal change. Attach a sign to indicate direction (up or right is positive, down or left is negative).	Rise and run are oriented (signed). Units are determined by graph increments (not blocks). The "rise over run" ratio and "run over rise" ratio are reciprocals.
Algebraic ratio (A)	Subtract *y*-coordinates for change in *y*. Subtract *x*-coordinates for change in *x*.	"Change" is oriented (signed). The "change in *y* over change in *x*" and "change in *x* over change in *y*" ratios are reciprocals.
Functional property (F)	Interchange the word slope with the phrase "rate of change".	Slope describes the coordinated change of two covarying quantities.
Parametric coefficient (PC)	Algebraically manipulate an equation into slope-intercept form or point-slope form. Identify the coefficient *m* of *x*.	The coefficient of *x* reveals different information depending on the form of the linear equation.
Real-world situation (R)	Identify the real-world quantity associated with the input and output variable (using any type of representation).	Interpret change as it relates to a real-world variable (i.e., a decrease in price shows depreciation over time).
Determining property (D)	Calculate the negative reciprocal. Recognize that equal slopes indicate two lines are parallel. Recognize that negative reciprocal slopes indicate two lines are perpendicular.	Slope indicates the number of points shared by two linear relationships and how they intersect (if at all).
Behavior indicator (B)	Visually determine if a line increases/ decreases.	An increasing (decreasing) relationship is one in which the variables change in the same (opposite) direction. MA2: A positive rate of change indicates two variables change in the same direction.
Linear constant (L)	Choose any two points on a graph/in a table when given multiple points.	Slope is independent of the points chosen since the ratio of change between the dependent and independent variables is constant.
Physical property (P)	Visually recognize a line's "steepness".	MA3: The rate of change indicates the amount of change in the dependent variable per unit change in the independent variable.

Limitation and Future Study

Limitations in the study design impact the ability to generalize our results beyond the students involved in this study. In particular, the concept of slope is generally taught beginning in the middle grades and our study focused only on college students. It would also be helpful to conduct a longitudinal study to gain more insight on how students' procedural proficiencies and conceptual underpinnings develop or persist.

There are three primary areas of research that scholars may wish to explore further. First, conducing a similar study with an expanded pool of participants from high schools and four-year colleges might provide additional or even different results. Second, repeated in-depth interviews over time with students or longitudinal studies involving pretests and repeated post-tests would provide more detailed insight on how students' procedural proficiencies and conceptual underpinnings develop or persist over time. Third, while we know that many college students lack a deep understanding of slope, what is still not well known is how college instructors are addressing the topics of slope with their students. Observations of college instructors' teaching lessons on slope may yield interesting data that could perhaps be useful to other mathematics instructors. What's more, a number of high school students take algebra and precalculus classes before they enter college. A study of both high school and college instructors could help point out similarities and differences in the methods for teaching slope and what points are emphasized in their instruction of this topic.

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