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Prospective Mathematics Teachers' Choice and Use of Representations in Teaching Limit Concept

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Abstract

The purpose of this case study was to examine prospective mathematics teachers' lessons in the context of choice and use of representations while teaching limit. The participants were four senior prospective mathematics teachers. The data were obtained from the participants' lessons plans, the video recordings of their lessons, and the semi-structured interviews. We categorized the representations used by the participants as real number line, tabular, figural, graphical, algebraic and verbal. While the participants mostly used the verbal representations during their lessons, they used real number line least. They used the graphical representations sometimes by means of mathematical software. The participants mostly used tabular representations in their first lessons to make an introduction to limit, however two participants also used them in their last lessons to make students comprehend the limits at infinity. They used the algebraic representations to express the limit algebraically, to present the properties of limit and to solve examples. They performed the conversions among the real number line, tabular, figural, graphical, algebraic and verbal representations. The conversions were mostly from the algebraic representation to the verbal representation and vice versa.

Introduction

Shulman (1987) explained pedagogical content knowledge (PCK) as the transformation of the subject matter knowledge (SMK) that is, presenting the subject in a more comprehensible way. The transformation requires the use of models, analogies, metaphors, examples, representations, presentations and simulations that can become a bridge between the teachers' understanding of the subject and the expected understanding of students (Uşak, 2005). Rowland, Huckstep, and Thwaites (2005) explained PCK in mathematics teaching as the teachers' transforming their knowledge into a form, which the learners can use, and their using different sources, representations or analogies to teach the mathematical thought. Ball, Lubinski, and Mewborn (2001) indicated that teachers needed not only to know the content conceptually but also to know the connections between ideas, and the necessary representations and the common misconceptions. Elia, Gagatsis, Panaoura, Zachariades, and Zoulinaki (2009) recommended making connections between topics, to use multiple representations and to make students involve in the process of constructing mathematical knowledge. Apart from this, Shulman (1986) stated that the most important element of PCK is to use representations in education (Turner, 2008).

Representations are any configuration that can denote, symbolize characters, images, concrete objects or otherwise represent something else (Palmer, 1978; Kaput, 1985; Goldin, 1987, 1998 cited in DeWindt-King & Goldin, 2003). Bruner (1974) stated that representations are important mediators in developing abstract understandings (cited in Rowland, Turner, Thwaites, and Huckstep, 2009). When looked from the point of mathematics, representations have played an important role in construction of mathematical knowledge. Duval (1999) emphasized that representations are crucial in understanding students' mathematical thinking and stated that there are three specific requirements in learning mathematics:

- to compare similar representations within the same register in order to discriminate relevant values within a mathematical understanding,
- to convert a representation from a register to another one,
- to discriminate the specific way of working in order to understand the mathematical processing that is performed in this register.

Dreyfus (1990) stated that learning process proceeds through four stages that are: (a) using a single representations (b) using more than one representations (c) making links between parallel representations (d) integrating and flexibly switching among representations. Using only one representation cannot fully reflect a mathematical construction as each representations has different advantages so using multiple representations for the same mathematical concept plays a central role in building mathematical understanding (Duval, 2002; Elia, Panaoura, Eracleous, & Gagatsis, 2007 cited in Elia et. al., 2009). Moru (2006) stated that understanding of limit could not be achieved without relating different representations. Kaput (1992) claimed that use of more than one representation in teaching help students to develop better understanding of the mathematical concepts. In the report of NCTM (2000), use of representations is recommended for a better mathematical thinking and is stated the importance of the use different representations as follows:

> Different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses (NCTM, 2000, p. 360).

With the combination of multiple representations, students are no longer constrained by the strengths and weaknesses of one particular representation (Elia et. al., 2009). For the acquisition of the concept is necessary understanding the same concept in multiple systems of representations, and the ability to use the representations and to convert one representation from one another. (Elia et. al., 2009). The importance of representations in mathematics teaching is handled in the framework specific to mathematics teaching named Knowledge Quartet (KQ) which helps to observe, develop and evaluate the mathematical knowledge and the knowledge of mathematics teaching together (Rowland, Huckstep, & Thwaites, 2005; Rowland et al., 2009). KQ is a very comprehensive framework in observing, assessing and developing the prospective mathematics teachers' SMK and PCK (Kula, 2011). The KQ has four unit called Foundation, Transformation, Connection, and Contingency (Rowland, Huckstep, & Thwaites, 2005). Transformation comprises choice and use of examples, use of instructional materials, choice and use of representations, and teacher demonstration.

This study constitutes the one part of a comprehensive study aiming to examine the prospective mathematics teachers' lessons in terms of codes of KQ. One of these codes was choice and use of representations. Because of the reasons that the each concept in the Calculus depends on limit in some way (Salas & Hille, 1990), and it is not possible to learn the concepts of continuity, derivative and integral without learning the concept of limit (cited in Bukova, 2006), this study focuses on the representations which are chosen and used in teaching limit.

Elia et al. (2009) examined 12th grade students' abilities in conversions between geometric and algebraic representations of limit and indicated that conceptual understanding of limit were occurred when students made conversions. They also stated that the limit concept linked the algebraic and geometric representations and these two representations were necessary to comprehend what the concept of limit is. However, Cornu (1991) indicated that one of the four major obstacles in history of the limit concept was the failure to link geometry with numbers and there was a problem to transfer from the geometrical figures to numerical interpretation of the limit concept. Domingos (2009) investigated the tertiary level students' difficulties in understanding of limit and stated when translating verbal representation into a symbolic representation, students' performances decreased significantly. He inferred that giving the graph of the function in teaching limit supports the conversion to algebraic representation and suggested geometrically interpreting the graph of the function firstly and then there should be a conversion to the algebraic representation. In a similar way, Hofe (1997) examined two students' epistemological obstacles in determining the limit of the rate of change of the function within a computer-aided learning environment, and asked for students first to observe the changes in the graphical representation by using mathematical software and then to find the limit by using the algebraic representation. Baştürk and Dönmez (2011) studied with 37 prospective mathematics teachers and draw attention to the necessity that students recognize the graphs, draw the graph of function by using its algebraic formula and interpret them. Duru (2011) analyzed 95 prospective mathematics teachers' conceptions about the limit of a function given in the form of both graphical and symbolic, differences of their graphical and algebraic performances and their misconceptions. He stated that the prospective teachers sometimes had problems in interpreting the limit of function by using the graphs however they were more successful in graphical representation compared to the algebraic representation. He expressed that the reason was that the graphs were effective visualizing tools and they made the invisible things visible (Arcavi, 2003; McCormik, De Fentim, & Brown, 1987 cited in Duru, 2011).

The verbal, real number line and tabular representations as well as graphical and algebraic representations are used for the limit concept (Elia et al. 2009; Karataş, Güven, & Çekmez, 2011; Kabael, Barak, & Özdaş, 2015). However, there are some problems regarding the use of some of these representations. Sezgin Memnun, Aydın, Özbilen and Erdoğan (2017) examined the abstraction processes of the two 12th-grade students through the RBC+C abstraction model and used the problems for abstracting limit knowledge. They asked students to approach any real number from the right or left for different events, form a list, and fill in the gaps in the given table by using the data in this list. They analyzed the participants' cognitive actions and found that they had some difficulties recognizing and building-with knowledge of real numbers. Domingos (2009) stated that students could expressed the limit verbally but they had some problems while expressing it algebraically. The reason may be that only one of the representations is used or different representations are used in an isolated way from each other in teaching limit. Özmantar and Yeşildere (2008) stated that using only algebraic representation in finding the limit of the function posed an obstacle for conceptual understanding and it leaded to not being able to make a connection between the limit and the operations carried out.

The representations have different functions and association between these representations helps to create a meaningful whole. The graphical representation should be used to show that there can be limit even at points where the function is not defined; apart from algebraic representation and the graphical representation should be given and associated to find the limit of piecewise function (Baştürk & Dönmez, 2011). Bergthold (1999) stated that examining the graph and table related to a function and finding the limit of a function in more than one way were two critical components in understanding the concept of limit (cited in Özmantar & Yeşildere, 2008). Domingos (2009) proposed that for the integration of the mathematical knowledge representations should be associated with each other. In a similar way, Baştürk and Dönmez (2011) stated that it was necessary to use and associate different representations with each other to show the subject from different perspectives. Apart from this, whether the limit value of the function was found correctly could also be tested by using different representations (Bergthold, 1999 cited in Özmantar & Yeşildere, 2008). The researchers consider that using representations was important to remove the obstacles and difficulties encountered in learning limit (Bukova, 2006; Hofe, 1997; Orton, 1983; Sanchez, 1996). Additionally, Karatas, Guven, and Cekmez (2011) stated that even though many researches about the concepts of limit and continuity, and the difficulties concerning these concepts were carried out (Cornu, 1991; Ferrini-Mundy & Graham, 1991; Mamona-Downs, 2001; Szydlik, 2000; Tall & Vinner, 1981; Davis & Vinner, 1986; Sierpinska, 1987), there have been few studies about the representations used to teach these concepts. In the studies in which representations of different mathematical concepts were examined (Elia et al. 2009; Hwang, Chen, Dung, & Yang, 2007; Karataş, Güven, & Çekmez, 2011; Özmantar, Akkoc, Bingolbali, Demir, & Ergene, 2010), predominantly the use of verbal, algebraic and graphical representations and the conversion between these representations was studied.

Significance of the Study

Even though there were studies about using representations in solving specific limit problems by teachers and prospective teachers, there could not be seen any research about how they choose and used representations in their teaching. Thus, it is considered to be important to study how the prospective teachers choice and use representations and how they effects their lessons. Which representations do prospective mathematics teachers prefer when planning their teaching? What directs their choices? Which references do they make use of when selecting the types of representations during the planning? Do they change the type of representations that they selected during planning while teaching? What are the reasons of these changes?

Purpose of the Study

Within the framework of the questions mentioned above, the purpose of the study was to examine prospective mathematics teachers' choice and use of representations in their teaching of limit.

Research Questions

The research questions were as follows:

(1) (a) Which representations do prospective mathematics teachers prefer when planning the teaching of limit?
(b) What are the factors affecting their selections?

(2) (a) How do prospective mathematics teachers use the representations involved in the lesson plans during their teaching?
(b) Do they change the selected representations during teaching?
(c) What are the factors that cause these changes?

Method

The study was designed as a qualitative case study aiming to analyze the participants' choice and use of representations in limit. Creswell (2003) states that the case study is used to analyze a program, an event, or a process in-depth analysis. In this study, we examined the choice of representation of the prospective mathematics teachers and whether they change their choices while teaching. If the participants have changed their choices, we addressed the reasons underlying these changes. In order to be able to investigate these changes in detail, we used a case study which allows answering what-why-how questions. In this study carried out with multiple case design, each prospective mathematics teachers was considered as a case. Thus, we analyzed four cases to understand the similarities and differences between the participants. For each case, the participants' own choices and uses of the representations was examined.

Participants

The participants were four senior secondary prospective mathematics teachers (three females and one male). They took the courses such as Calculus, Analytic Geometry, Discrete Mathematics, Differential Equations, Linear Algebra, Mathematical Modeling, Mathematical Problem Solving, Mathematical Software, Introduction to Educational Sciences, Curriculum Development, Teaching Methods in Mathematics, Examination of Mathematics Textbooks, etc. Apart from this, there were the courses related to school-based placement named School Experience I- II, and Teaching Practice. During this process, the participants had the chance to improve their knowledge about the concept of limit in the Calculus I-II courses. At the same time, in Teaching Methods in Mathematics I-II, they had also chance to learn how to teach this concept. Even though the representations used for the teaching of the limit concept had not been directly handled in these courses, the presentation of various mathematical concepts via different representations was handled.

The participants took part in the research voluntarily. The real names of the participants were not used in the study; the pseudonyms (Deniz, Umay, Caner, Alev) which they chose were used while giving information about them and while the data were being analyzed. There were about 13-15 secondary school students in the participants' lessons. In addition, the real name of their students did not used in the study. Generally, the students start secondary school at the age of 15 and graduate at the age of 18. According to the national secondary school mathematics curriculum, the concept of limit introduced for the first time in the senior year of the secondary school. Before teaching the concept of limit, the students thought the topics such as functions, sequences, absolute value, etc. It was determined that the formal definition of limit should not be given in our national curriculum and it was suggested to intuitively describe limit (Bergthold, 1999; Cornu, 1991 cited Özmantar & Yeşildere, 2008). In our national mathematics curriculum (MEB, 2006), the objectives and the proposed representations related to the concept of limit were given as follows:

Table 1. The objectives of the limit concept in the curriculum

The objectives	Representations*
1. He/she explains an independent variable's approach to a given number with examples.	
Showing the approach to the number 6 from left and right using tables	T
Showing the approach from the left and right to a point on the real number line corresponding to the real number a selected	N
Verbally explaining that the approach to the number from the right is by decreasing values and that the approach to the number from the left is by increasing values	V
2. By explaining the limit from left and the limit from right of a function at a point with examples, He/she indicates the relationship between the limit of a function at one point and its limit from left and from right.	
Examining the limit of a function at a given point (ex. $f(x) = 2x + 1$)	T-V-A-G
$f(x) = x / \|x\|$ for the determination of the limit of a an unidentifiable function	T-A-G
Associating with the circle the increase of the number of regular polygons corners of which passing from geometric approach to algebraic approach are located on the circle	F -V-A
3. He/she indicates the limit law (when finding the limit of a sum, difference, product, quotient, etc.) and performs applications about it.	
Calculating the limit of the summation, difference, multiplication and division of two functions	A
Calculating limits for absolute value, logarithmic and exponential functions	A
Expressing Sandwich theorem without proof	A
4. He/she performs applications related to the limits of piecewise functions and absolute value function.	

For the function $f:R\rightarrow R,\ f(x)=\begin{cases}3x+1, & x<2\\ 1-4x^2, & 2\le x<4\\ 5x+3, & x\ge 4\end{cases}$ examination of limits at $x=2,\ x=3$ ve $x=4$	A
For the function $f:R\rightarrow R,\ f(x)=\dfrac{x^2-1}{\lvert x-1\rvert}$ examination of limits at $x=-1$ ve $x=1$	A
5. He/she indicates the set of extended real numbers and explains on the graph the limit for infinity and infinite limit concepts in the real variable and real valued functions.	
For the function *f*(*x*) = $\dfrac{1}{x}$ examination of $\lim\limits_{x\to 0^-}\dfrac{1}{x}$, $\lim\limits_{x\to 0^+}\dfrac{1}{x}$, $\lim\limits_{x\to -\infty}\dfrac{1}{x}$ and $\lim\limits_{x\to +\infty}\dfrac{1}{x}$	A-G-V-A-V-A
6. He/she indicates the features related to the limit of trigonometric functions	
Determination of the limit values for the x values of various trigonometric functions	A

* The abbreviations of the representations were A: algebraic, F: figural, G: graphical, N: number line, T: tabular, V: verbal.

Prior to their teaching, the participants observed their mentors' lessons. Afterwards they taught themselves and no intervention was made during this period. In other words, the mentors had not helped the participants in any way as to the method they should use in planning and conducting their lessons.

Data Collection

The data obtained from the participants' lesson plans, 16-hour video records of their lessons (four hours per participant) and voice records of the semi-structured interviews (seven interviews per participant). With the use of multiple data collection methods in case studies will increase the effectiveness of the study and reduce the probable bias of the researcher on the study (Hartley, 1995). For this reason, the study was triangulated using observation, interview and documents. The data sources and their intended use were given as follows:

Lesson plans: The participants prepared their lesson plans before teaching. These lesson plans were used to determine which representations the participants had chosen for limit teaching and whether they used any representation that was different from the ones, which were exemplified in the curriculum. We interviewed participants about their lesson plans. From the lesson plans, we also determined whether the participants changed their choices during their teaching. Thus, the lesson plans were the first data collection tool seeking answers to our first research question.

Video records of the lessons: The participants' lessons were recorded. These video records were used to observe how the participants used the representations in their lesson plans throughout their teaching. With these records, we aimed to investigate whether the participants used the representations in lesson plans exactly in their teaching and whether they used other representations. We used the video recordings for constructing the interviews' questions. We compared each lesson plans with the lessons and then, made interviews. For further analysis, we transcribed the video recordings after the all lessons were completed. The transcripts included the expressions of the prospective teachers and the students, the screenshots of the presentations and videos projected on the screen and the writings on the board. We had the opportunity investigate the second research question by using video recordings.

Voice records of the semi-structured interviews: We made interviews to determine the participants' choices of representations and their opinions about how they used them in their lessons. The first interview was called "The Interview before the Lessons" and included questions about their lesson plans. The following four interviews were made after each lesson of the participants and named after the relevant lesson such as "The Interview after the First Lesson". These four interviews were made to collect the participants' views on the choice and use of representations. Also a general interview named "General Interview about the Lessons" was made after the participants completed whole lessons. The final interview named "Interview about the KQ's codes" was made for the participants to evaluate themselves regarding the KQ's codes. However, it was only focused on the views about the choice and use of representations in this study. Through interviews, we answered the first and the second research questions.

The Roles of the Researchers

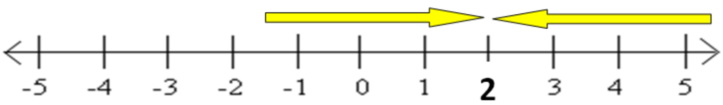
In the study, the first researcher was in the classroom to observe the lessons. Thus, the researcher has become familiar with the data for first hand. She set up the video camera before the lesson and found it near the camera to check the camera's shooting area. She performed zoom-in or zoom-out operations when necessary. The researcher became a non-participant observer and did not intervene lessons in any way. She took observation notes during the lesson. These observation notes were used in the interviews after the lessons. In all the data analysis process, two researchers coded the data independently from each other and then they came together and discussed their analysis and if it was needed they made consensus.

Data Analysis

Table 1 was used while analyzing the lesson plans and the representations included in the participants’ lesson plans were determined. While analyzing the representations usage, we used the thematic coding. For this reason, we examined the literature in terms of the representations used for mathematics especially limit teaching and we constructed a code list. Goldin (2000) asserted that the representations included spoken and written symbols, static figural models and pictures, manipulative models, and real world situations. Afterwards, the representations used for Calculus have been examined and the rule of four model frequently used was encountered. The Calculus Consortium based at Harvard University, established a multi-representational approach to calculus named The Rule of Four: geometrically, numerically, analytically, and verbally (Kennedy, 1997 cited in Schwalbach & Dosemagen, 2000). The rule of four where mathematics topics are introduced geometrically, numerically, analytically, and verbally (Aspinwall, 1996; 1997) extended from the rule of three (Hughes-Hallet et al., 1994) because of the recognizing of the importance of verbal representation. It has been determined that these representations were also taken into account in studies on teaching limit.

For example, while Güçler (2013) took into account the written words, graphs, and symbolic representation, Elia et al. (2009) has taken into account the geometric and algebraic representations, however graphs were also included in geometric representations. The usage of written words of Güçler (2013) was handled as verbal representation in other studies. It was also observed that real number line and tabular representations were used for the limit teaching (Karataş, Güven, & Çekmez, 2011). In the frame of the examined literature, the representations used by the participants were gathered under certain categories named as real number line, tabular, figural, graphical, algebraic and verbal in this study. The representations considered in data analysis explained below:

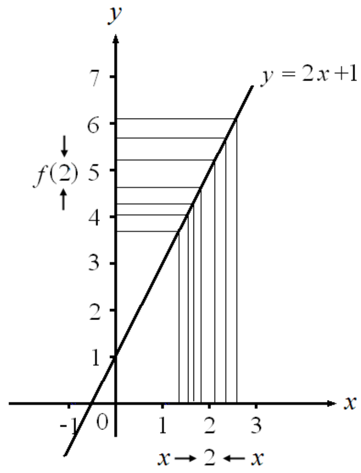
1. Number line representation includes the use of the number line for showing the approach to a number. For instance number line representation can be used to suggest the approach to the number 2 from the right and left.



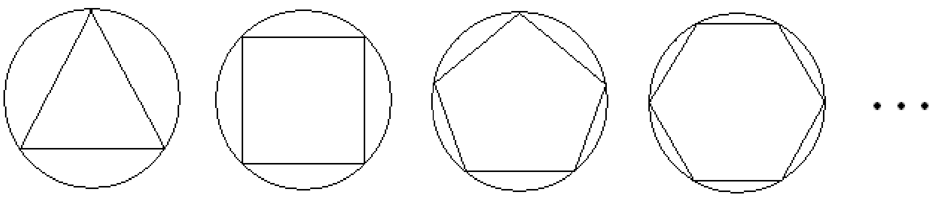
2. Tabular representation includes tables used to determine the approach of the values of a function for various points and to determine the existence of a limit. For example; the table below can be used for the $f: IR \rightarrow IR, f(x) = 2x + 1$ function.

						2					
x	1,9	1,95	1,99	1,999	1,9999		2,0001	2,001	2,01	2,05	2,1
$f(x)$	4,8	4,9	4,98	4,998	4,9998		5,0002	5,002	5,02	5,1	5,2

3. Graphical representation includes function graphics used to determine the value that the function takes for various points and to determine the existence of a limit. For example; the graphical representation for the $f: IR \rightarrow IR, f(x) = 2x + 1$ function has been given below.



4. Algebraic representation includes algebraic representations regarding the concept of limit. Since the focus is on the concept of limit for this study; algebraic representations without the limit term have not been taken into account. An example that can be used for the algebraic representation to examine the limit of the $f: IR \rightarrow IR, f(x) = 2x + 1$ function $at\ x = 2$ has been given below.
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x+1) = 5$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(2x+1) = 5$$
$$\Rightarrow \lim_{x\to 2} f(x) = 5$$
5. Verbal representation only includes the verbal explanations regarding limit and limit related concepts. For example; verbal representation given below can be used to express the algebraic representation of $\lim_{x\to 2} f(x) = 5$.
The limit of the function as x approaches 2 is 5.
6. Figural representation includes the pictures, graphs and video representations used to form the idea of limit in one's mind. For example, the concept of approach can be suggested to the students by a figural representation showing that the polygon sides of which are on the circle. Increasing the number of sides of regular polygons inscribed in a circle so that the perimeters of the regular polygons approach the perimeter of the circle.



The frequency of using these representations was given in tables and the representations labeled as N for number line, T for tabular, F for figural, G for graphical, A for algebraic, and V for verbal. For instance while indicating how many times a graphical representation was used in a lesson, the number of the different graphs were taken into consideration to avoid repetition. Apart from this, for instance as the situations in which the graphical and algebraic representations were used by being associated were included in the table related to conversion, they were also included in the tables that indicated how many times the graphical and algebraic representations were used.

When conversions between different representations were discussed, the labels and the dashes, such as A-G-V were used. This means that first an algebraic representation was used, then it was converted to a graphical representation, and finally, the graphical representation was converted to a verbal representation. The transcriptions of interviews and lessons were used to support the findings obtained from the lesson plans and lessons of the participants related to their choice and use of representations. We asked the participants to review the transcripts and thus we received the participants' confirmations. While presenting findings, we supported the transcripts of the lesson plans, video recordings and interviews. We provided the triangulation of data and analyzed the data by reaching consensus as two researchers. Thus, it was aimed to increase the dependability of the work (Lincoln & Guba, 1985; Merriam, 1995).

Results

Choice of Representations

While answering the first research question stated as “Which representations do prospective mathematics teachers prefer when planning the teaching of limit? What are the factors affecting their selections?”, the representations chose by participants to achieve the objectives in their lesson plans were analyzed and the findings were given in Table 2.

Table 2 The Choice of Representations: frequencies by the participant and the objectives of limit

Objectives	Deniz	Umay	Caner	Alev
1.	5 times F 1 times T-N-A	4 times F	1 times G	3 times F 1 times V-A
2.	2 times T-G- A 4 times G-A 1 times A-V	1 times F 6 times G 2 times A 3 times G-A 1 times G-T-A 1 times N-V-A-T	2 times A 2 times G-A 1 times G-T 1 times T-A 1 times G-T-A	2 times F 6 times G 1 times A 1 times G-T
3.	15 times A	10 times A	17 times A	9 times A
4.	4 times G 2 times A	2 times G 1 times A 1 times G-A	7 times A	4 times A 1 times G 1 times A-V
5.	4 times A 1 times F 1 times N 3 times G-A 1 times G-T-A	3 times F 13 times A 4 times T-A 2 times G-A	9 times A 1 times G-A	4 times A 1 times V-A
6.	8 times A 3 times G-A 1 times A-G	1 times F 8 times A 2 times G-A	15 times A	1 times F 4 times A 1 times A-G

The participants first examined the national mathematics curriculum and considered the suggestions of the curriculum while preparing their lesson plans. The participants stated that they made use of textbooks for some considerations such as how to present the limit. The participants mostly preferred algebraic representation in their lesson plans and they did not include much verbal representations. While participants who used verbal representations allocated a space for the verbal representations in their lesson plans in order to indicate how the limit of a function along with expanded real numbers could be expressed. Apart from this, they also extended the tabular representation which was exemplified at the beginning of the subject and only in the first two objectives in the curriculum to different objectives. In addition, the number line and the figural representation which was exemplified only once in the curriculum was used by the participants in different objectives. The participants did not include the representations only took part in the curriculum to their lesson plans, but also added some representations based on their own decisions and expressed this as follows:

> First I paid attention to the objectives. Then I thought how I could achieve them… I examined different textbooks and also a book chapter on obstacles and misconceptions of students while learning the limit concept. (Deniz, The Interview before the Lessons)

> First I took into consideration the objectives in the curriculum. Moreover, looking at the curriculum, I decided what I should make the students gain. I used the exercises given in the curriculum to make the objectives learned. I thought about how these activities would contribute to the process of learning. I thought whether I need to do different activities or need to use the curriculum activities directly. In this direction, I made a draft of the lesson plan in my mind. Apart from this, I tried to uncover the things that are about the concept of limit but not included in the curriculum and which I thought the students should definitely learn. (Caner, The Interview before the Lessons)

> I thought about teaching the limit in all aspects by making changes on the graphs of different functions with the help of mathematical software when necessary. I used various sources such as textbooks and internet sources. These researches helped me see my deficiencies and I added them to my plan. I looked at the textbook to understand how the things I talked about limit were explained and found a few important points and added them to the plan. (Alev, The Interview before the Lessons)

The curriculum mainly includes algebraic and graphical representations and the participants as well used them more than the other representations in their lesson plans (see Table 2). The participants stated that they considered to choose particularly graphical representations while teaching limit:

> In the introduction to limit I am planning to talk about the concept of approaching by using real number axis. I will use graphs on the analytic plane. The students may think that finding the limit is like finding the value of function, to prevent this I am planning to use more emphasis on the graph. (Deniz, The Interview before the Lessons)

> Because I decided to make the plan by adding more visual aspects. I mean with the help of mathematical software, by making some changes on the graphs of different functions when necessary, I show this to the students in different ways. (Alev, The Interview before the Lessons)

In addition to the curriculum, Deniz made use of various textbooks and another book which has a chapter on misconceptions specific to the concept of limit in order to prevent the misconceptions. Deniz made the students watch a flash animation to enable them to understand the approaching. Similarly, Umay made use of various figural representations even though the figurals have been included only once in the curriculum in a similar situation. She integrated these into the lesson. In the interview, Umay stated that she will make use of animations for teaching.

> I believe that teaching and learning the concept of limit is difficult. That is why I wanted to give real life examples. But since I couldn't find any similar examples in the text book, I decided to prepare them myself. I took the chasm animation that I had watched. Using Paint program, I prepared an animation including the approach to the movie star and established or non-established character examples. (Umay, The Interview before the Lessons)

The participants integrated the only one example regarding figural representation in the curriculum that is the increase of the sides of a regular polygon drawn inside the circle with sides touching the circle but not explained how to use this figural representation in the curriculum. The participants except Caner included this example in their lesson plans.

Deniz stated that she preferred to increase the number of example of graphs independent of textbooks and the curriculum. Umay pointed out that she would draw the graph of the function using the software Derive which was not included in the curriculum and was used for limit calculation of absolute value functions:

> When I examined the curriculum and text books I saw that there were not enough graphs. So I tried to add many graphs. (Deniz, The Interview before the Lessons)

> I thought that the curriculum was insufficient in suggesting the use of algebraic representation only for the absolute value function. I thought of drawing the graph of the function via software and graphically approaching the points for which limits are calculated. (Umay, The Interview before the Lessons)

Use of Representations

While answering the second research question stated as "How do prospective mathematics teachers use the representations involved in the lesson plans during their teaching?", "Do they change the selected representations during teaching?", "What are the factors that cause these changes?", the participants' use of representations in their lessons were analyzed. The representations used by the participants were real number line, tabular, figural, graphical, algebraic and verbal. The findings regarding the use of real number line representations were given in Table 3.

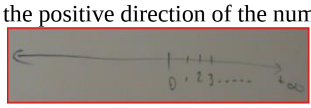
Table 3 The Use of Real Number Line Representations: frequencies by the participant and lesson

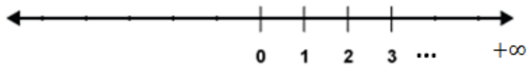
Participant	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Deniz	2	-	-	2
Umay	1	-	-	-
Caner	-	-	-	2
Alev	-	1	-	-

They rarely used real number line for visualization in their lessons. Alev used the real number line to express the approach to a number from left and right when she was solving a problem even though it was not included in the lesson plan. Deniz and Umay used the real number line when they were making introduction to the concept of limit. Deniz also used the real number line to make an introduction to the extended real number set. Even though it was not included in the lesson plan, Caner used the number line in his fourth lesson to enable one of his students to grasp $\pm\infty$ following the question asked.

Caner: Now, this is my real number line, right? I am lining up the numbers on the number line. I am moving towards the positive direction of the number line. What comes here, the last? Plus infinity, right?

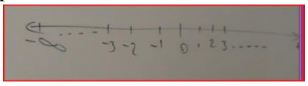
Board:

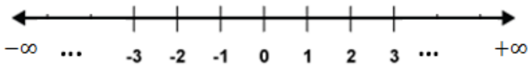




Caner: I mean when we look at the number line, we see that all the numbers that are on the right of 0 are called positive numbers. And the numbers that are on the left of 0 are…

Board:





The findings about the tabular representations were given in Table 4. All participants used the tabular representations in their first lessons to introduce the limit concept. In addition, Deniz and Umay used the tabular representations in their fourth lessons about limit at infinity.

Table 4. The Use of Tabular Representations: frequencies by the participant and lesson

Participant	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Deniz	2	-	-	1
Umay	2	-	-	1
Caner	3	-	-	-
Alev	1	-	-	-

Umay explained the reason why she used tabular representation in teaching limit as follows:

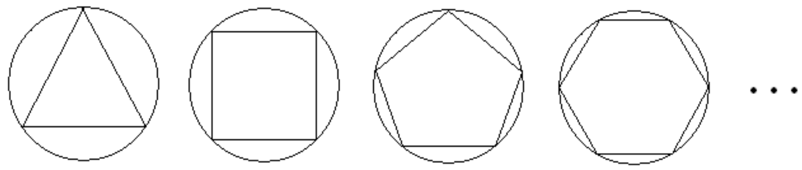
> I used tables for some concepts like approaching, approaching to infinity and to zero. Because as different values in x and y are given together on the table, approaching to the point looks regular. So the value of the limit can be found more easily. (Umay, Interview about the KQ's codes)

The findings regarding the use of the figural representations were given in Table5. The participants generally used the figural representations in their first and fourth lessons. All participants used the figural representations in their first lessons to introduce the limit concept and in their fourth lessons about limit at infinity. The participants used the figural representations to visualize the ideas of the approach, limit, infinity, etc.

Table 5. The Use of Figural Representations: frequencies by the participant and lesson

Participant	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Deniz	5	-	-	2
Umay	5	-	4	2
Caner	2	-	-	2
Alev	5	-	1	1

During their first lessons, Caner and Alev used the figural representation given below showing the fact that a regular polygon drawn inside a circle with sides touching the circle approaches a circle as the number of its sides increases. Caner used this example even though it was not included in his lesson plans.

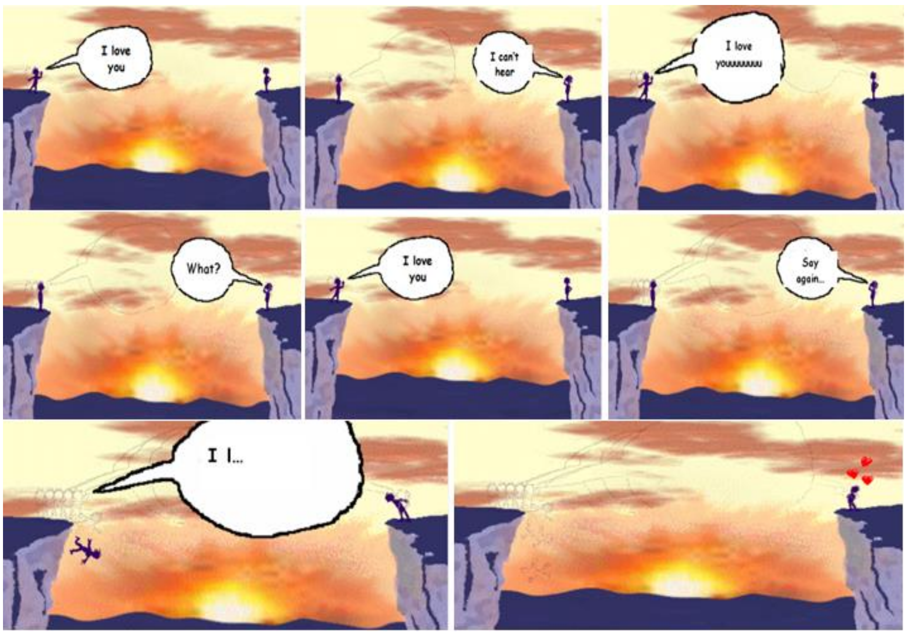


During the interview, Caner stated the reason for adding the figural representation to the teaching as follows:

> There was nothing related with the historical development of the concept of limit in the curriculum and thus I did not add it to my lesson plan. However, thinking that it might attract the interest of students, I mentioned Achilles Paradox and the reason why people might need the concept of limit in the past. I tried to explain that even though it was not known how to calculate the area of a circle, it might still be calculated if one knows how to calculate the area of polygons and that the concept of limit arose out of a need. (Caner, The Interview after the First Lesson)

In the first lesson, Umay used an animation to make her students perceive the concept of approaching from the right and left hand. Umay made students watch the animation about getting closer to the edge of a cliff mentioned below.

Slide:



Umay asked a question in her second lesson “what do you think about where the limit is used in daily life”. For this question, students’ expressions such as “computers have a certain memory and they cannot pass it”; “Factories. They determine the daily limit. For example there is a limit for goods to be produced daily.”; “It is necessary to pass over 50 to be successful in the exam.” showed that they considered limit as a boundary not to be passed and a maximum value. Also, Alev gave some examples which may cause students to perceive the concept of limit as a boundary not to be passed. Some of these examples were credit card limit, speed limit and alcohol limit. To eliminate these misconceptions emerged in Alev and Umay’s lessons it may be plausible to mention the difference between the meaning of limit in daily language and its meaning in mathematics or to give some examples which may avoid limit to be seen as a boundary not to be passed. Students’ examples like “speed limit is 50 km per hour and this speed cannot be passed” can be clarified by saying “In a road where the speed limit is 50, the speed of the car might be 45 or 55; when this limit is exceeded its speed could be slowed down to 50 km per hour” and thus student would understand it is possible reach to the speed limit which is 50 km per hour from the right or the left.

The findings regarding the use of graphical representations were given in Table 6. The participants generally used the graphical representations.

Table 6. The Use of Graphical Representations: frequencies by the participant and lesson

Participant	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Deniz	6	3	2	8
Umay	6	4	4	3
Caner	9	1	-	2
Alev	5	-	5	-

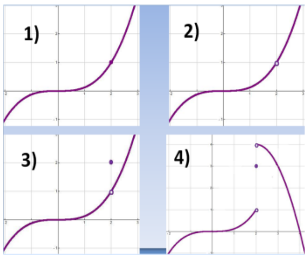
They sometimes drew the graphs via the mathematical software and sometimes drew arrows and dots themselves on the board to show the approaching a point on the graph from left and right hand. They used graphical representations to visualize the concept and to make students understand the subject better. Deniz and Alev expressed the reasons why they used the graphical representations as follows:

> With the use of the mathematics software, I am planning to teach limit to students in every aspect by making changes on the graphs of different functions when it is necessary. (Alev, The General Interview about the Lessons)

> I think that it is more important to use graphs while forming the concept of limit. Because it is the best way to visualize it. (Deniz, The General Interview about the Lessons)

Deniz used the graphs that she drew on the board and the ones she put on the slides. But she did not use any software to draw the graphs. For example in the first lesson, she used the graphical representations to discuss the approaches from the left and right hand and to make a connection between these approaches and the limit of function. In the four graphs that Deniz chose and used, that the function was define/undefined at x=2, limit from the right and left hand and the existence of limit stood out. Hereby with the graphs, it was aimed to make conclusions like that there may be limit at a point where the function was undefined, there may not be limit at points where it was defined and the equation of the limit from right and left led to the existence of limit.

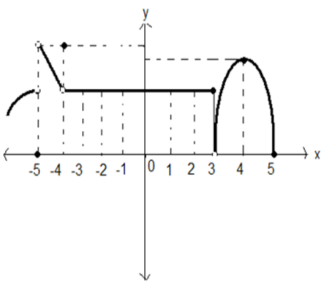
Slide:



Deniz: Now, you discuss it with each other for four minutes. Again as groups. You remember all our findings, right? If you want, I can show it again. You can ask questions. We can conclude from here that there are graphs which are connected to each other but they are defined in different ways. What do we conclude? You discuss it with each other. (*15 seconds later*) I will ask you one by one. I am expecting good answers.

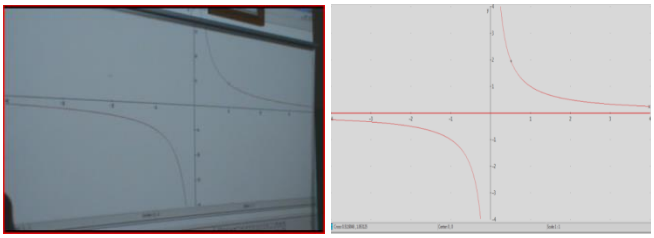
Umay used the graphical representations in her all lessons. She used the graphs of functions by drawing them in mathematical software, by showing in her slides and by drawing on the board. For instance, in her first lesson Umay discussed with the students whether the function whose graph was given had a limit with any whole number value between [-5, 5] by using the following graph. On this single graph, Umay discussed the limit cases with her students while Deniz discussed the same things on her four graphs.

Slide:



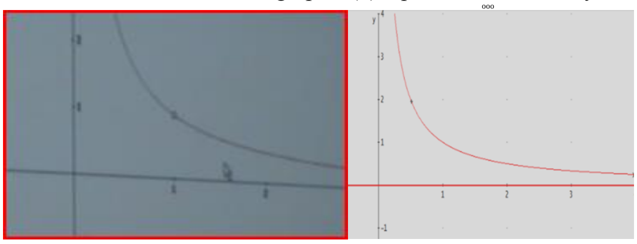
Umay showed her students by taking advantages of the dynamics of the software (DERIVE), which values of the function approached when x=0 was approached from left and right. This graph helped the students find the limit especially due to the dynamic structure of the software and due to that the change in y in response to x that is selected on the graph is observed and it drew the attention of the students. Umay also used this approach exemplified for the change in x=1 in her lesson for different x values.

Slide:



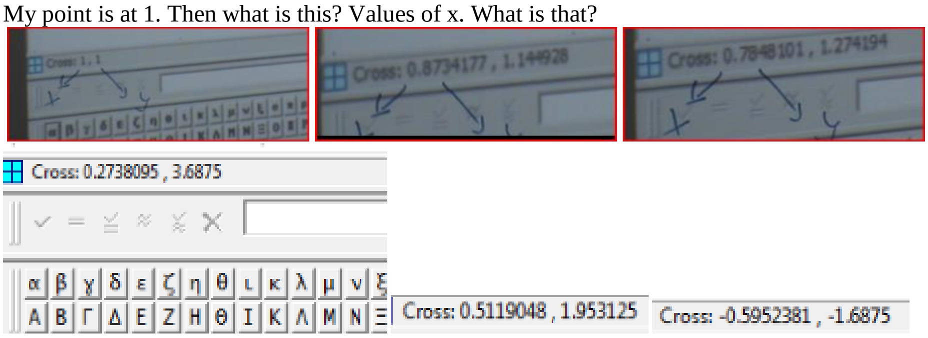
Umay: Yes friends, let's look at the graph, f(x) equals to 1 divided by x. Now on this graph.

Slide:



Umay: My point is at 1. Then what is this? Values of x. What is that?

Board:



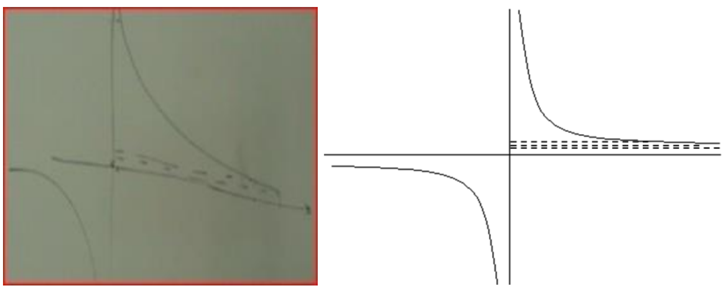
Umay: Values of y. Now which direction should I take this point to? I should approach 0 slowly. Can you see the values?

Umay explained the reasons why she used software like this while drawing the graphs and giving different x values as follows:

> Secondly I wanted to make the concretization in a better way. I wanted them to see the points exactly and what is cutting what and where. That is why I used it. (Umay, The Interview after the Fourth Lesson)

Caner preferred to draw the graphs on the board. For example, in his fourth lesson, he used the graphical representation to find limits at infinity and by using this representation, he made it easier for students to find limit and helped them visualize it. Caner first drew the graph of the function and asked the students to find some limit values for this function. When the students gave different answers, he did the drawing, which Umay did on DERIVE, on the graph to help students find the correct answer.

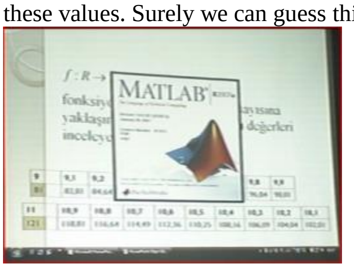
Board:



Alev took advantage of dynamics of MATLAB in her first lesson. By using MATLAB she presented an example showing which point the values of the function approached while the function was approaching a given x value from left and right. In this presentation, Alev showed not only how to draw the graph of the function but also which value the function approached when the values of x approached a point from left and right on the graph.

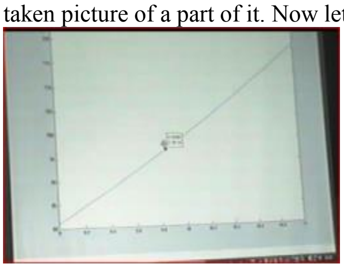
Alev: Let the x^2 function which is defined with real numbers and let's see the value the function will have while approaching 10. If we approach from 9 to 10 and from 11 to 10, we all know somewhat that they will take these values. Surely we can guess this.

Slide:



Alev: Let's look at this. I should tell you this. I tried it on this software but I cannot show x-y coordinates. You know that the ones that are above are y and the ones that are below are x. I will be glad if you think as if I had taken picture of a part of it. Now let's think that I want to approach 10 and I take value from here.

Slide:



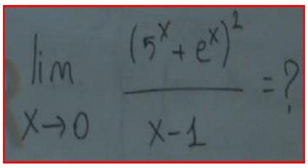
The findings about the algebraic representations of the limit concept were given in Table 7. The participants used the algebraic representations of the limit concept very commonly. While Caner used algebraic representations the most, Alev used them the least.

Table 7 The Use of Algebraic Representations: frequencies by the participant and lesson

Participant	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Deniz	15	19	7	11
Umay	7	16	5	18
Caner	20	27	14	18
Alev	7	11	14	12

The participants generally used the algebraic representation while making conversions between different representations. Besides, there were times when they only used algebraic representation. For instance, as it happened in the third lesson of Deniz, the participants used the algebraic representation while finding the limits of different functions. As it is hard to draw the graph of the function without using any software, it is considered that in this kind of examples, they preferred algebraic representation.

Board:



$$\lim_{x\to 0}\frac{(5^x+e^x)^2}{x-1}$$

The findings about the participants' use of verbal representations were given in Table 8. While Caner used the verbal representations the most, Alev used them the least.

Table 8 The Use of Verbal Representations: frequencies by the participant and lesson

Participant	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Deniz	11	20	9	26
Umay	20	22	5	22
Caner	25	23	33	22
Alev	9	9	10	6

Caner was very careful while using the verbal representations of limit, he rarely used verbal expressions wrongly. Deniz and Umay used the verbal representations of limit almost at the same frequency. Alev had the most difficulties to express mathematical expressions verbally. It is considered that the reason why Alev made mistakes while expressing the mathematical expressions verbally could be her limited subject matter knowledge as she stated in the interview before the lessons.

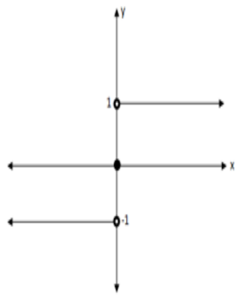
> To tell the truth, I don't think that my knowledge of limit is enough. I have many deficiencies. Actually I have been doing something to make up my deficiencies; I read things about this subject or I try to come across with different questions etc. because I think that I have many deficiencies both in my content knowledge and teaching. But I think that from time to time I will see them more clearly after some exercises and improve them in the course of time (Alev-Interview Before the Lessons).

In her first lesson, Alev gaved the function f:IR-{0}→IR,f(x)=|x|/x and asked her students to find the limit of the function at x=0. Two of her students gave the answers +∞ and undefined and three students gave the answer that limit from right hand was 1 and limit from left hand was -1. Besides, she could not get an answer from other students but in accordance with the answer of her three students, she expressed that there was no limit. In this example, while Alev was saying that there was no limit, she should have made conversions between different representations by using tabular, graphical and algebraic representation instead of using only verbal representation. So that, she could have shown her students that there were other ways to find the limit.

Slide:

x	f(x)
0,1	1
0,01	1
0,001	1
0,0001	1
0,00001	1
$x \to 0^+$	$f(x) \to 1$

x	f(x)
-0,1	-1
-0,01	-1
-0,001	-1
-0,0001	-1
-0,00001	-1
$x \to 0^-$	$f(x) \to -1$



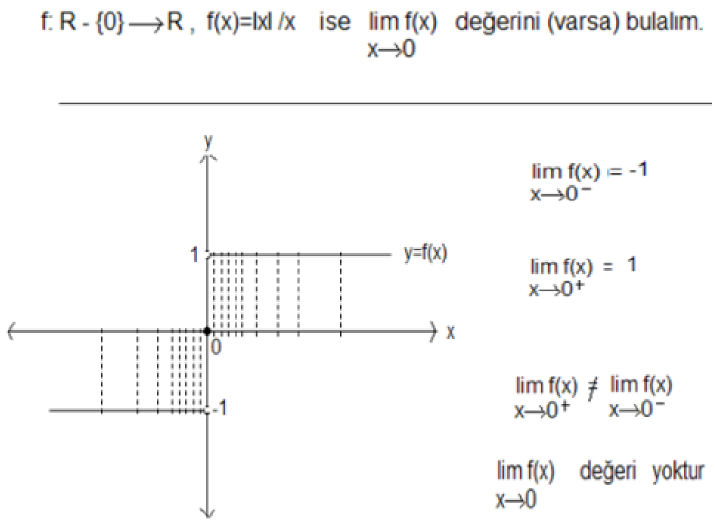
$$f(x) = |x| = \begin{cases} x\,;\ x \geq 0 \\ x\,;\ x < 0 \end{cases}$$

$$\lim_{x\to0^+} \frac{|x|}{x} = \lim_{x\to0^+} \frac{x}{x} = \lim_{x\to0^+} 1 = 1$$

$$\lim_{x\to0^-} \frac{|x|}{x} = \lim_{x\to0^-} \frac{-x}{x} = \lim_{x\to0^-} -1 = -1$$

Umay used the same example that Alev used, unlike Alev she made conversions between three representations which were verbal, graphical and algebraic while discussing the answer of the question. Moreover as it is given in the following slide, by using the term "neighborhood", Umay also examined the change of the function in the neighborhood of 0 on the graph.

Slide:



It is important to make conversions between different representations to make limit understood. In their lessons, the participants converted different types of representations into each other to present the students different representations of limit and to make them comprehend limit better (see Table 9).

Table 9 The Use of Conversion between Representations: frequencies by the participant and lesson

Participant	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Deniz	2 times (G-V-A-A-A) 1 times (G-V-A-A-A-V) 1 times (G-V-A-A-V-A) 1 times (F-N-T-N-V-A-A) 1 times (A-V-T-G-V-A-V)	12 times (A-V) 1 times (G-V) 1 times (G-A) 1 times (V-A) 1 times (G-V-V)	2 times (A-V) 1 times (G-A-A-A)	2 times (V-A) 1 times (A-V) 1 times (G-V) 1 times (F-A-V) 1 times (N-V-V) 1 times (G-V-A) 2 times (G-V-A-V) 1 times (G-A-V-T-V) 1 times (G-A-V-A-G-V)
Umay	2 times (A-V) 1 times (F-V) 1 times (G-V-V) 1 times (A-V-G) 1 times (A-G-V) 1 times (G-A-V-V) 1 times (F-N-V-A-T) 1 times (G-V-V-V-T-V-A)	9 times (A-V) 1 times (V-V) 1 times (G-A) 3 times (A-V-V)	2 times (F-V) 1 times (A-G) 1 times (G-A) 1 times (F-A-V) 1 times (G-G-A-V)	8 times (A-V) 1 times (G-A-V) 1 times (F-A-G-V-V-V) 1 times (A-G-T-V-V-V-V-A)
Caner	5 times (V-A) 2 times (F-A) 1 times (G-A) 1 times (G-V-A) 1 times (G-A-V-V) 1 times (G-V-A-V-A) 1 times (G-T-V-V-A) 1 times (G-T-V-V-A-V-A-V) 1 times (V-A-A-V-G-V-A-V-A-V-A)	1 times (V-A-V) 15 times (A-V) 1 times (A-V-A-V) 1 times (A-V-A)	3 times (V-A) 3 times (A-V) 1 times (A-V-V-A-V)	10 times (V-A) 2 times(A-V) 2 times (F-A) 1 times (A-N) 1 times (G-V) 1 times (G-A-V)
Alev	5 times(A-V) 3 times(F-V) 1 times (T-G) 1 times (G-V)	2 times(A-V) 1 times (V-A-N)	4 times (A-V) 1 times (G-V) 1 times (F-A-V-G) 1 times (A-G-G-A)	4 times (V-A) 2 times (A-V) 1 times (A-F-V)

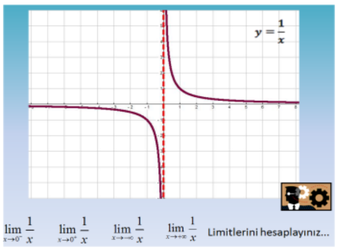
The participants made conversions between different representations. Caner made conversions between representations most and Alev the least. They tried to imply the concept of limit by using multiple representations more in their first lessons. Especially it is important to use different representations and link them in the first two lessons to construct the concept. For instance while Caner was mainly making conversions between the verbal and algebraic representations, he used graphical, tabular, algebraic and verbal representations in his first lesson and made the connection between them and explained the reason as follows:

> I intended to introduce the limit by starting the lesson with graphs, making students comprehend approach from left and right on the graph and to talk about limit first with the help of graph. I mean to make students understand limit better I used graphs, tables, symbols, lines and curves more than enough. (Caner, Interview about the KQ's codes)

In additionally, the participants generally used to converse the figural representations in their first lesson for students to perceive the idea of limit, in their second or third lessons in which the participants talked about the properties of limit and solved examples, they generally made conversions between two representations; predominantly algebraic and verbal. For instance, while the participants were presenting the students the properties of limit, generally they preferred to give them algebraically first and then express them verbally. In their fourth lessons in which they made an introduction to the set of extended real numbers, while mostly they made conversions between two representations, they also made conversion between four different representations. While the participants were making conversions between representations, they generally preferred to make the conversions themselves rather than asking the students' ideas. For example, the participants did not ask the students to write algebraically what they had expressed verbally on their notebooks or on the board. They also did not ask for verbal, algebraic or graphical representations after showing the values that the function approached while approaching a given point on the table. The participants made conversions between algebraic and verbal representations the most. The conversions from algebraic to verbal representations were more than the conversions from verbal to algebraic representations.

In Deniz's fourth lesson, she gave the graph of the function $f: IR - \{0\} \rightarrow IR,\ f(x) = \frac{1}{x}$ and asked to find the limits of the function while it approached x=0 from left and right hand and while the values of x went to plus-minus infinity and then she showed these values using tabular representations.

Slide:



Deniz: Then let's continue. The graph of the function $f(x) = \frac{1}{x}$ was given. Here we will try to find the limits when the values of x approach 0 from left and right and also when the values of x go to plus-minus infinity. Now think about it. Is there anyone who wants to talk? For example while approaching 0 from left and right. What do you think about that?

ooo

Slide:

x	f(x)	x	f(x)	x	f(x)	x	f(x)
0,1	10	-0,1	-10	10	0,1	-10	-0,1
0,01	100	-0,01	-100	100	0,01	-100	-0,01
0,001	1000	-0,001	-1000	1000	0,001	-1000	-0,001
0,0001	10000	-0,0001	-10000	10000	0,0001	-10000	-0,0001
0,00001	100000	-0,00001	-100000	100000	0,00001	-100000	-0,00001
...	...	...	...	...	...	...	...
$x \to 0^+$	$f(x) \to \infty$	$x \to 0^-$	$f(x) \to -\infty$	$x \to \infty$	$f(x) \to 0^+$	$x \to -\infty$	$f(x) \to 0^-$

Deniz: Now here I took some values but we don't have to analyze them separately. What are the values while the values of x approach 0 or go to infinity? In this case, we can write these values now. Because we saw it.

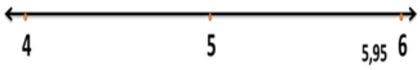
Deniz expressed that she tried to use multiple representations and she used the graphs and the tables especially to make students understand limit intuitively:

> I used graphs and tables a lot especially to make students understand limit better and visualize it in their minds. Moreover I tried to associate their uses with each other. (Deniz, Interview about the KQ's codes)

Deniz used the conversion from real number line to tabular representations. In her first lesson when approaching 5, she used firstly number line and after converted things what she explained to the tabular representation.

Deniz: 5.95, for example. Let's even show this. How do we show it? (*She draws a number line on the board.*)

Board:



ooo

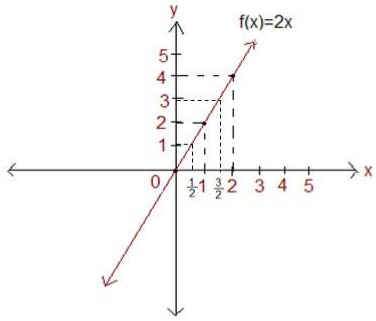
Deniz: Ok. Now, if we show what we have found on a table…

Slide:

A	B
4	6
4,5	5,5
4,9	5,1
4,99	5,01
4,999	5,001
4,99999	5,00001
...	...

Umay made conversions between four representations; graphical, tabular, verbal and algebraic for two different situations. For instance, in her first lesson, Umay first drew the graph of the function $f(x) = 2x$ and showed the approach to $x = 1$ from left and right hand on the graph and indicated that the function had the same value in both approaches; then presented this approach by using the table and in the final stage expressed algebraically that limit is 2.

Slide:



Umay: The graph of the function f(x)=2x is given. Here, I want to find limit by approaching the value of 1, the value of x=1 umm from left and right. What do you think it is?

ooo

Umay: I approached 1 by making the values smaller than 1 go up and up and up. And again by taking the values smaller than 1 and making them go down, I approached 2 umm 1. Umm I see that in both conditions the value approached 2. So, I say that limit is 2. Let's see the table.

Slide:

→ 1 ←

x	1	0,75	0,8	0,9	0,99	0,999	1,001	1,01	1,1	1,2	1,25	1,5
f(x)	2	1,5	1,6	1,8	1.98	1,998	2,002	2,02	2,2	2,4	2,5	3

ooo

Umay: As limits are equal from left and right, limit is 2. How can we write it symbolically?

Slide:

$\lim_{x \to 1^-} f(x) = 2$ → $\lim_{x \to 1} f(x) = 2$

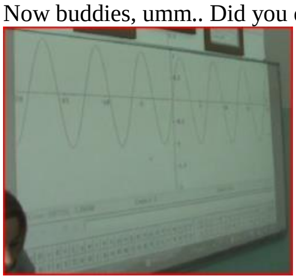
$\lim_{x \to 1^+} f(x) = 2$ ↗

Alev made conversion between different representations the least. The conversion from tabular to graphical representations was made once. Alev presented students the table which showed the changes of the values of y while the function $f(x)=x^2$ was approaching x=10 from x=9 and x=11 and then by showing the graph of this function she helped the students understand these approaches on the graph of the function. Alev made conversions by using at most three representations. For example, in her third lesson, Alev converted algebraic representation to graphical by using DERIVE to draw the graphs of trigonometric functions. However she did not show any particular approach to a point on the graph and expected the students to think about the approach from left and right hand by looking at the graphs. Alev also showed the limit values which the trigonometric functions determined algebraically by drawing their graphs.

Slide: $\lim_{x \to 0} \cos x = 1 \quad \lim_{x \to 0} \sin x = 0$

Alev: Now buddies, umm.. Did you draw the graphs of the cosine and the sine? Do you know? Before.

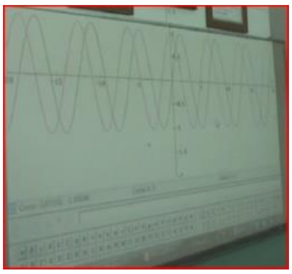
Slide:



Alev: Is it right?

Student: Yes. It is.

Slide:



Alev: Then is this right, too? Ok. It is clearer now, right? Which one is sine and which one is cosine?

Conclusion and Discussion

The participants chose and used mainly six types of representations for the concept of limit that were real number line, tabular, figural, graphical, algebraic, and verbal. The participants generally used the representation in their lesson plans. When preparing their lesson plans, the participants mainly examined national mathematics curriculum and its examples and some textbooks. When they thought it was insufficient, they considered to use multiple representations and to convert them. They thought adding more graphs/examples and including videos/animations that could attract the students' attention while linking the concept with daily life. Their own learning experiences have affected their choice of representation type.

The representations that the participants had chosen affected the way they used representations during the lessons. In their lessons, the participants generally used the same representations that they had chosen before. While generally the algebraic representation was in the foreground, they mostly preferred the verbal and algebraic representations while teaching limit. Besides, among the number line, tabular, figural, graphical, algebraic and verbal representations that were recommended in the literature to be used in teaching the concept of limit, the one which was used the least by the participants were the number line and the tabular representations. The participants who did not include verbal representations a lot in their lesson plans used verbal representations in their lessons very often. Besides, the participants used algebraic representations more than they had stated in their lesson plans. The participants added new representation types when faced with questions from students, when students had difficulties and misconceptions or when they thought it was to make the lessons more interesting.

The participants used the real number line representation (a) to introduce the concept of limit, (b) to introduce the extended real number set, (c) to express the approach to a number from left and right hand, and (d) to answer a student question. Even though it was not included in the lesson plans, one participant used the number line representation to show the approach from the right and left hand and one participant used the number line representation to be able to answer the question of a student. Similarly, the participants used the tabular representation (a) to introduce the limit concept, (b) to mention neighborhood along with the graphics, (c) to examine y values corresponding to changes of x values, and (c) to introduce the limit at infinity. The participants did not use any tabular representations in their teaching except for the tabular representations included in the lesson plans. The participants used figural representation (a) to introduce the limit concept, (b) to relate limit concept with the real world, (c) to overcome the misconceptions about limit concept, (d) to introduce limit at infinity and (e) to visualize the ideas of the approach, limit, infinity, etc. They have generally preferred to use the figural representations they specified in the lesson plans; however, they also made use of this representation that they thought would attract attention during their teaching.

The participants considered that the graphical representations was important in teaching the concept of limit and they paid attention to choose and use graphs. They used the graphical representation (a) to introduce the limit concept, (b) to express the approach from left and right, (c) to explain the neighborhood of points on the graphs, (d) to introduce limit at infinity, (e) to compute the limit of a function, and (f) to show a particular theorem or law. The participants drew attention to "defined-undefined statements", "limit from left and right hand" and "the existence of limit" in various graphs and they tried to form the limit conceptually. However in some cases when it was hard for the participants to draw graphs, it was observed that they used only algebraic representations. In such cases, mathematical software can be used and along with the algebraic expression, graphs and tables can be examined. Kepçeoğlu and Yavuz (2017) pointed out that teaching limit using GeoGebra software is superior to the achievement of student teachers in terms of limit compared to traditional teaching. Also even though they did not make a formal definition of limit while they were using the graphs of the functions, they paid attention to the neighborhood of points on the graphs. Such an approach on the graph enables making a formal definition prospectively (Domingos, 2009). Domingos (2009) and Hofe (1997) stated that in teaching limit, giving the graph of the limit first supported the conversion to algebraic representation. In a similar way Elia, et al. (2009) stated that algebraic and graphical representations of limit were necessary to comprehend the concept of limit. Similarly, the participants tried to make the students understand the approach from the right and left hand by using graphs after which they have passed onto algebraic representation. The participants used algebraic representation (a) to express the approach to a number from left and right, (b) to express the approach to from left and right, (c) to write and solve an example, (d) to answer a student's question, (e) to compute the limit of a function, and (f) to show a particular theorem or law. Even though the participants did not give much importance to the algebraic representations according to the other representations in their lesson plans, it was one of the most frequently used representation forms. When the participants were teaching the limit laws, they preferred the algebraic representation as is suggested in the curriculum, however different to the curriculum they made some changes by increasing the number of examples. The participants mostly made use of verbal representation in lessons even though they almost did not include it in their lesson plans. They used the verbal representation (a) to express the approach to a number from left and right, (b) to express the approach to from left and right, (c) to solve an example, (d) to answer a student question, (e) to express a particular theorem or law, (f) to give an extra explanation for one representation and (g) to conversion between the representations.

The participants made conversions between two representations, mainly between algebraic and verbal representations and sometimes between four and five representations. This approach was thought to be important. Even though the participants made the conversions between the representations themselves, that they made conversions between the representations in their lessons will help the students express their ideas by using different representations. The participants made conversions between representations mostly in their first lessons when they were forming of limit. This approach of the participants are considered to be important because using only algebraic representation while finding the limit of a function can hinder the conceptual understanding and it can lead to not being able to make a connection between the limit and the operations carried out (Özmantar & Yeşildere, 2008). In addition, making conversions between representations is important as it shows that limit can be found in more than one way. Bergthold (1999) stated that finding the value of limit in more than one way had critical importance (cited in Özmantar & Yeşildere, 2008). When considered from this point of view, choosing the representations carefully, considering which representation should be used for which objective and which conversions should be made becomes crucial. The participants switched between two representation types in the second and third lessons while making their students carried out examples, while they switched between more than two representation types in the first lessons during which they made an introduction to the concept of limit and during in the final lessons where they introduced the concept of expanded real numbers. Bergthold (1999) stated that examining the graphical and tabular values of the functions were important to understand limit (cited in Özmantar & Yeşildere, 2008). As a matter of fact some researchers (Dunham & Osborne, 1991; Knuth, 2000; McCoy, 1994; Schoenfeld, Smith, & Arcavi, 1993) recommended encouraging students to link tabular and graphical representations to algebraic representations (cited in You, 2006). However it was stated in the studies that the teachers had difficulty in making connections among algebraic, tabular and graphical representations (Even, 1990; Norman, 1992; Stein et al., 1990, see in Rider, 2004). Ferrini-Mundy and Graham (1989) observed that when the students were asked to find the limit of the function at the given point, the students were quite successful but when they were asked questions about the geometric interpretation of the limit, they interpreted the algebraic and graphical representations separately and they could not make any connection between them. Even (1990) stated that secondary prospective teachers were unable to make connections between algebraic and graphical representations. Contrary to Even (1990), the participants made connections between these two representations.

Karatas, Guven and Cekmez (2011) studied the ways how the prospective teachers defined the verbal, algebraic and graphical representations of the concepts of limit and continuity. In the consequence of the study, it was stated that using multiple representations like verbal, graphical and algebraic representations in designing learning activities could be important in examining the limit of a function at points that it is defined or undefined. In parallel with this suggestion, it is considered that the fact that the participants used different representations while designing their teaching supported the understanding of the concept of limit.

Ball (1990) stated that many teachers lack the knowledge of representations. So it becomes important whether the teachers have the knowledge of representations and are able to use these representations in their lessons. Stein et al. (1990) and Wilson (1994) state that teachers need to re-learn the content of what they teach and to learn to use multiple representations and to increase their content knowledge of connections among different representations (cited in You, 2006). Özaltun (2014) also examined the usage of representations in the context of teachers' knowledge of student thinking and recommended using different representations and their conversion for conceptual understanding. In this direction, we recommended to the prospective teachers both integrate their knowledge about the subject and their knowledge about the representations and reflect their teaching.

It was observed that the representation types used by the participants had some reflections on their students. Like the participants, their students also mostly used the verbal representation followed by the algebraic representation and switched between them. None of the students made use of number line, tabular and figural representation whereas one student who solved a question on the board preferred to draw the graph of the function. The fact that the student preferred to draw the graph even though he/she was not asked to may be due to the fact that the participant used graphic representation more during the course.

One participant caused a misconception to occur when he/she used a game while trying to make the students understand the fact the function did not have to be defined at the point for which the limit is examined. The reflection of the emphasis of “we can never be that point” on the students is formed like “if a function was defined at a point, it cannot have limit at that point. Within the misconceptions regarding the value of limit for never being reachable, it is emphasized that the limit value could never be reached (Szydlik, 2000; Williams, 1989, 2001). The participant realized this misconception during the activity to determine the limit on the graphs, after which he/she has given an example from daily life even though it was not included in the course plan and has tried to overcome this misconception by using both the verbal and algebraic representation types.

The participants’ teaching could cause misconceptions to arise because they were sometimes not very careful when using verbal representations and because they did not make the required explanations about figural representations. Students developed various preconceptions about the concept of limit since the participants did not properly explain examples such as credit card limit, speed limit and alcohol limit and these preconceptions have had some reflections on the course environment. Because of this misconception, the limit concept is perceived as the maximum value to be reached and as a boundary not to be exceeded in the daily language (Cornu, 1991; Davis & Vinner, 1986; Szydlik, 2000; Tall & Schwarzenberger, 1978; Williams, 1989) and it is stated that from this aspect it contradicts with mathematical limit (Özmantar & Yeşildere, 2008). Similarly, when the algebraic representation were used in a wrong way, the students repeated them. To this end, it could be stated that the selection of the correct representation type is as important as its usage in the learning environment.

Implementations

The participants benefited from the observations and notes in getting to know their students, getting used to the learning environment etc.. The researchers spent too much time to transcribe the data accurately during the study. The videos were watched twice after the first transcription in order to prevent data loss and required corrections were made. This contributed to the researchers regarding the familiarity with the context of participants and the data. The participants selected the representation types themselves and no intervention was made during this process. Interviews were carried out with the participants both before the lessons regarding representation type selection and after the lessons regarding the use of selected representations in order to determine why they made that selection and how they used the selected representation type. In addition, only lessons’ videos were used to determine the reasons for the selection and usage by the participants of the representation type for students and this has caused some problems. It is thought that carrying out interviews with students after the lessons regarding the use of representation types would contribute to the determination of these reasons and their reflections. In addition, it is thought that carrying out the detailed interviews regarding the representations used by the participants and students are important in understanding how these representations affect learning and putting forth the difficulties and conveniences of the relevant representation types. Future studies could be designed by keeping these factors in mind. Whereas it was helpful during the study to examine the lesson plans and lessons of four different participants for seeing the differences, it was possible for the researchers to miss various points. It is suggested that carrying out future studies with a single prospective mathematics teacher, increasing the number of lessons and collecting data from the students in these lessons. To carry out deeper observations and to support the study with prospective teachers’ and students’ ideas may be more beneficial. In addition, the representation selection of the prospective mathematics teacher according to the subject and its reflections on the students may also be examined.

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