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Logical-Mathematical Constructions in an Initial Course at the University: A View of Their Syntactic, Semantic and Pragmatic Aspects

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Abstract

We present an analysis of students' formal constructions in mathematics regarding to syntactic, semantic and pragmatic aspects. The analyzed tasks correspond to students of the Course of Mathematics for the admission to the university. Our study was qualitative, consisted in the identification, analysis and interpretation, focused in logic features, of the students' oral and written responses. Here we refer particularly to the following formal constructions: a) the decision of if an object belongs or not to a defined class, b) the determination of the value of truth of a composed proposition, c) immediate inference, d) analytic deduction and e) the refutation.

Key words: Formal construction; Refutation; Deductive inference; Analytic deduction; Mathematics learning; Admission to the university

Introduction

This is a qualitative study that consists in the identification, analysis and interpretation, focused in logic resources, of the students' oral and written responses used in structuring their arguments to ensure the validity of the mathematical knowledge brought into play in the classroom (Falsetti, Alvarez, 2011). For this work we rely on the perspective for the logical analysis of mathematical students productions presented in Durand-Guerrier work (2004, 2005, 2008). The author studied the mathematical students' constructions under the syntactic, semantic and pragmatic point of view. She focused on pre-calculus constructions with logic of first order. For our part, we adapt this point of view for studying the productions of students in order to see more clearly their logical aspects. We selected some learning activities that were proposed for a regular course at the university initial mathematics course where secondary school contents are "revisited". We analyze the responses of students from a general initial course of mathematics of the National University of General Sarmiento sited in Buenos Aires province, South America, Argentina. We consider that the didactical contribution of our work is that we identify formal constructions proposed in learning activities and then we analyze the constructions under a pragmatic, semantic and syntactic perspective in order to clarify the manifestation of the logic reasoning in mathematical constructions and procedures.

The Context Where the Study Was Done

The Course of University Readiness

The National University of General Sarmiento is a university sited in the suburb of Buenos Aires City. Every applicant to this university must approve a preparatory course of mathematics. The mathematics subject deals with content of the secondary level college: real numbers, algebra, notions of geometry and numerical functions. The classroom work is developed through activities proposed in a printed guide for all students' groups. The educational proposal considers that a student learns mathematical when it is able to perform "mathematical activity" on mathematical problems. That is, he is able to draw up a plan of action, propose conjectures, test and sustain with a reasoned discourse the procedure used and the validity of proposed conjectures. In the classroom, the teacher proposes to solve problems promoting individual work or into small groups work. He manages the development of the work and then directs a collective participation for exposing the approaches proposed by the groups that are compared and justified. Students recognize that they have studied the contents of the course before, but they say that the kind of work in the classroom at this course is very different to the secondary

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school because they are accustomed to repeat schema and apply mechanisms (Falsetti, Cesaratto & Marino, 2011, 2012).

Students are on average between 18 and 30 years old. They come from different schools in the area of influence of the University. Their previous training experience is very heterogeneous, some students of the admission course are studying the last year of high school, others finished the secondary level long times ago and others attended the secondary school for adults. In general, students attend classes and work therein with the activities proposed, however very few of them make their homework.

Theoretical Framework

In scientific mathematics the way of validating the production, i.e. ensuring that the results conform to requirements fixed by the community of scientific practice, has rigorous formats (Godino & Recio, 2001) involving logical deductive reasoning. For this reason, proof and proving are essential subjects in mathematics education, such as the proceedings of the International Committee of Mathematical Instruction (ICMI, 2009) reveal. Understanding and acquisition of logical rules and the use of specific symbols in proof and proving are, in our opinion, the pillars of the formal aspect of the mathematical reasoning. In order to study both components, logic and symbol, in the students' productions, we introduce the notion of *formal construction* to refer to:

a) A discursive or symbolic elaboration (with symbols or mathematical semiotic registers (Duval, 2001) where highlights some "logical construction" it is "mode of constructing, expressing and linking thoughts (and parts of thinking) with different specific content" (Rosental & Iudin, 1965),
b) A procedure that includes symbolic manipulation on a mathematical expression through mechanisms supported by valid definitions and properties in some semantic model in which the given expression it makes sense,
c) A development resulting from the combination of points a) and b).

An example that illustrates what is said in point b) is the following: to decide if two algebraic expressions are equivalent, a possible formal construction is an equation resulting by matching them, followed by an algebraic manipulation to obtain the solution and subsequent interpretation of the solutions' set. In this case, if this set is the reference one, the equivalence of the expressions can be concluded. Another example of formal construction would be the succession of algebraic transformations, from which is derived the formula for calculation of the roots of a quadratic expression.

In the formal constructions, the semantic context or model has a central role. This is the theoretical and practical mathematics reference, composed by content: objects, properties, rules, postulates, procedures, etc., organized by means of scientific methods and epistemological criteria throughout the history of the discipline construction. From what it is said above, we conclude that formal constructions have the following characteristics:

a) They may expressed in one, or more, mathematical semiotic registers (tables, algebraic symbols, Cartesian diagram, etc.);
b) They may involve some cognitive function associated with the use of registers (Duval, 2001): the formation of the representation, the translation of a representation to another, the conversion of a register to another or treatment in the same register;
c) They suppose some logical construction in the sense that was defined before;
d) For analytical deductions, they are subject to the semantic model that gives meaning to the expressions on which deductions are made. For example, given the expression "$ax=b$", the deduction of the solutions of this equation is not the same if the semantic context is the set of real numbers, with the usual operations, that if the semantic context is the set of matrices of real coefficients with matrix operations.

To identify the formal constructions that we identified in the students responses, we rely on the work of Campistrous (1993), about the learning logical procedures and on the works of Valdes (1989) and Gutierrez (2006), both on elementary logic and natural deduction systems. Campistrous, relying on Jean Piaget research, determines that the development of thinking is associated to the domain of logical procedures with logical constructions of thought, which are classified into: concepts, judgements and reasoning. According to this author, the procedures associated with judgments are: to identify structures, to analyse if it meets conditions of veracity, to assign a truth value, to transform and to modify judgments. The procedures associated with reasoning consist of: immediate inferences, deductions for separation, refutation, direct demonstration, indirect demonstration, argumentation, inferences that are based on syllogism and reductive inferences. Reductive inferences are obtained by analogy, induction or other forms not deductive. In relation to logic, we highlight in this work to the formal constructions more promoted by the learning activities: the recognition of membership in a class, the determination of truth of composed propositions, deductive inferences and the refutations.

The Recognition of Membership in a Class

The decision of if an object belongs or not to a defined class not includes necessarily deductive logical rules, it can be leaded by analogy or induction. This entails the comprehension of the scope of the properties or definitions that determine the class. When this recognition is made deductively, we say that it shares characteristics of a deductive inference as we explain later. Campistrous (1993) notes this formal construction as a conceptual procedure because is an elemental procedure that comprise rational acts as analysis, synthesis, comparison, concretion, abstraction, generalization, particularization for isolating properties or associating properties to an object.

The Recognition of the Truth Condition of a Composed Proposition

A proposition is composed when it results from combining other propositions by logical connectors such that conjunction ($\wedge$), inclusive disjunction ($\vee$), implication ($\rightarrow$) and denial ($\neg$). This combination is made under certain syntax rules (for example: conjunction connector is binary). There are other logical connectors but these can be expressed as a combination of the four presented. In this work the determination of the truth of a composed proposition is considered as the result of the reflective understanding of the role of each statement and the prepositions in the composed one, in a close relationship with the semantic context. That is not the result of a manipulation of the truth values according to a truth table. According to Campistrous (1993), this formal construction corresponds to the judgement's treatment and it comprises rational actions as identifying structures or assigning truth value.

Deductive Inference

An immediate inference is a conclusion stemming from the application of a rule of inference of the "system of natural deduction (SND)" (Valdes, 1989; Gutierrez, 2006). This system consists of eight basic rules grouped into rules of implication, conjunction, disjunction and negation. Between these logical constructions, we particularize the recognition of a membership of a class as an immediate inference of implication. Knowing that if an object has certain characteristics belongs to a class, for a given object that has these characteristics then it belongs to the class defined by them. A deductive chain is obtained from a concatenation of the rules of the SND.

Analytic Deduction

It is the logical construction linked to point b) detailed above. It is a set of ordered and hierarchical steps comprising the symbolic manipulation of an expression that has meaning in the semantic context of reference. Such manipulation adheres to the syntactic rules of formation of mathematical expressions and is supported, as well as management and hierarchy of steps, by mathematical theory (definitions, properties, etc.) which forms the semantic context. As a result of the analytic deduction, a new symbolic expression is expected. An interpretation or significance of this expression can be done in the semantic context. The interpretation is free of imprecision (as it may occur in numerical manipulation).

The Refutation

The refutations can be analyzed by the basic rules of inference however we want to give a particular entity because of the importance in the mathematical reasoning. This term include: the counterexample, the recognition of the contradiction and the reductio ad absurdum. The counterexample is an example that proves that a statement is false. The recognition of the contradiction is when individual realizes that in a same discourse referred to the same matter there are two statements that say opposite things. A refutation also occurs when it is assumed that the consequence of an implication is true and from it, through logical deductive rules, a new conclusion is obtained. If this conclusion leads to a contradiction with other true proposition or with the antecedent of the implication, the refutation is by the reductio ad absurdum. According to Campistrous (1993) the last three constructions correspond to reasoning.

Criteria for the Analysis of the Formal Constructions

The intention of this work is a didactic study of formal constructions. In order to analyze them we consider the work of Durand-Guerrier (2004, 2005, 2008) who holds the hypothesis that to study the logical mathematics field, in a didactical sense, is necessary to consider the semantic, syntactic and pragmatic aspects. Many other authors adopted similar perspective but focused only in syntactic and semantic aspects. For Weber and Alcock (2004, in Knapp 2005) the semantic reasoning is when "the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" The syntactic reasoning is when the students produce a proof by manipulating definitions and facts in a "logically permissible way". Easdown (2009) defines syntactic reasoning as the superficial, which operates on the symbols without a reflection deep about what they mean in the context. While semantic reasoning is that relates to intuition, the acquisition or the experience. In his work he collects and analyses a series of specimens submitted to the students in situations of teaching, in which evidence of errors arising from the disconnection between the two types of reasoning.

For our study we will consider that syntactic aspect refers to how mathematical statements of formulas are built according to language rules, in this case the mathematical ones. The sequence and hierarchy of the terms in statements are guided by this aspect. Semantic aspect, however, refers to the values of truth in reference with a domain or mathematical field and the use and scope of quantifiers. Semantics is responsible for the domain objects, properties, relations, connectors and involved quantifiers and the relevant interpretation of the satisfaction of a formula for an assignment in a structure. In the words of Tarski (1944):

> Semantics is a discipline which, speaking loosely, deals with certain relation between expressions of a language and the objects (or "states of affairs") "refers to" by those expressions. As typical examples of semantic concepts we may mention the concept of designation, satisfaction and definition (p.345).

For the semantic treatment in a logical construction, it is necessary to specify the domain to which objects that are assigned to the symbolic variables of the sentences belong. It is also necessary to specify how variables are assigned and the predicates that involve properties or relations. Pragmatics refers to the individual's context, i.e. situation and personal mathematical knowledge of individuals that deal with a formal construction.

For example, the sentence $\forall x(x \times a = a \times x \wedge a \neq 0 \wedge a \neq 1)$ is constructed according to the syntax rules of mathematical language. In the domain, or semantic model, of real numbers, whereas the symbol $\times$ as the usual product of numbers, the meaning of this sentence is a particular case of the commutative property of real numbers with the product, and therefore is a universal true sentence. On the other hand, if the domain is the square real matrices of size $n \times n$, from the semantic point of view, whereas the symbol $\times$ as the usual product of matrices, $a \neq 0$ means the no null matrix and the 1 means the identity matrix, for a particular and adequate assigned *a,* the sentence is not true as the product of matrices is not always commutative. The pragmatic aspect in this case is explained as follows: analysis of the value of truth in the matrix algebra domain would be exhausted if it exhibits an example where such equality is not verified. But also you could search all cases where the equality $x \times a = a \times x$, for a given and fixed matrix *a,* is verified, in order to see that it defines a set strictly included in the domain and therefore the sentence is not universally true in it. In summary, searching cases for which the sentence is true as a means to analyze the initial sentence, may be more illustrative than the counterexample. As we see, both are different ways and this is what has to do with the pragmatic aspect, and of both the value of truth of the initial judgement can be inferred. As expressed, presented between formal constructions, analytic deduction would result from dialogue between the syntactic, semantic and pragmatic aspects of a series of inferences involving mathematical content.

Methodology of the Study

We worked with two class groups of approximately 20 students each, in 90 hours, for four months. As we presented above, students are young men and women, from 18 at 25 years old that finished their secondary school and attend an introductory and selective course of mathematics in order to be admitted at the university. The diagnostic tests show that they are not used to argue, explain, prove or demonstrate in mathematics classrooms. They also did not have any class of formal logic behind this course. Contents of the course were: general definition and representations of numerical function and linear, quadratic, polynomial, exponential, and logarithmic functions. We placed our attention to identify and analyze the more usual formal constructions manifested in the pre-university mathematics, in the subjects: linear, polynomial, and exponential functions.

The methodology was qualitative, based on data obtained by: a) direct and participant observation of class groups registered in spreadsheets, b) protocols of some students written productions (homework and exams).

To guide the observation, for each class was carried out a careful planning and a previous analysis of learning activities trying to anticipate the formal constructions that students could manifest with those ones. When necessary, the proposed activities were modified. In this way we anticipate some behavior to elaborate the spreadsheets. The later analysis was to confront the manifested behaviors in the class with the expected ones and to interpret them following the theoretical perspective. In the follow, to illustrate the way we did the participant observation, we present an extracting of an activity and then an extracting of the spreadsheet to collect data referring to it (Table 1).

For the function: $f:R \rightarrow R/f(x)=mx$,

a) *Write in symbols the image by f of the variable x_1.*

b) *What is the analytical development to verify that: "the image of the sum of two values is the sum of the images of each one."? In the problem of currencies' conversion, find the calculations in which you work with this property.*

c) *What is the analytical development to verify that: "the image of the product of a number and a variable is the product of the number and the image of the variable."? In the problem of currencies' conversion, find the calculations in which you work with this property.*

In items b) and c) not only a symbolic translation was expected but an exploration, numerical or algebraic, that lead to the analytic development.

Table 1. Logic Aspects and Analyses

Logic aspects to observe	Quantitative analysis		Qualitative analysis	Interpretations
Symbolic generalization and manipulation in order to deduce the properties of the proportional function.	Present students in the class. 20	Students who responded item a) 4; item b) 2; item c) 20.	4 (four) students responded item a) by writing by symbols the image of x_1 as $f(x_1)$. Another student responded b) by writing $f(x_1+x_2)=f(x_1)+f(x_2)$. Another one wrote $f(x_1+x_2)=m(x_1+x_2)$ but not continued with the deduction. When the teacher explained the resolution by taking up the individual responses, two expressions were presented in the blackboard for students choice the correct. All group of students observed that by means distributivity from the second expression it was possible to arrive to the first. Then all students wrote $f(kx)=m(kx)=k(mx)$ and they identified the expression in the previous activity.	Difficulty in writing a symbolic expression and evaluating it was observed. From symbolic schemas, students identified the properties in the currency conversion problem. Although they could not manipulate symbolic formulas.

Analysis of Some Formal Constructions Under the Syntactic, Semantic and Pragmatic Aspects

We present in this section the analysis of the more illustrative student's formal constructions by identifying semantic, syntactic and pragmatic aspects in each particularized formal constructions. We extract those samples from different sources: classes or written productions. According to the classification of Campistrous (1993), the first three sections are related to reasoning, the fourth is related to the judgement and the last has to see with concepts. For the first three sections we have meaningful data from the group of students of the class (40 students), showing particular behaviors of students that illustrate the semantic, syntactic and pragmatic aspects. About judgement and concepts, we analyze particular case from particular protocols because we did not find meaningful data to analyze those aspects in the class.

The Recognition of Membership in a Class

We illustrate this formal construction with an activity about second degree function. Students use to recognize such function with the format: $ax^2 + bx + c$, and the aim of the following statement is to identify a function of second degree when it is done with another expression.
Statement:

> *The function $f:R \rightarrow R$ whose expression is $f(x)=a(x-h)^2+k$ (with a, h and k real, $a \neq 0$) does correspond to a quadratic function? Why?*

In the class, students that reply correctly about the belonging to class, make the following two procedures: Some students (34%) raised the equation $a(x-h)^2+k=ax^2+bx+c$. They applied distributive property to the first member and then identified the coefficients obtained with the expression parameters *b* and *c*. Again the equality appears as a decision criterion. In this case raises an expression given to the formula ax^2+bx+c with prefixed letters equal. There is a very static idea about the expression of a quadratic function, it seems that what characterizes it is the location of those letters: *a*, *b* and *c*, and no others, attached to the powers of *x*. From the syntactic point of view, students pose an equality for finding conditions about the letters. The semantic aspect appears in the, implicit, criterion of equality of polynomials used when, after applying distributive property, they assign a symbolic expression to the coefficients *b* and *c*.

Others students (26%) assigned numerical values to parameters *h* and *k* and then developed the expression to compare it to the polynomial form given in class. By assigning numerical values to the parameters *h* and *k* manifests the pragmatic aspect by reducing a general expression to a particular case which represents all elements of the same class. Another 10% said that it was a quadratic function because its graph was a parabola, even without having plotting.

In the following particular case, is interesting to observe the manifestation of syntactic and semantic aspects when the student states that the object belongs to the class but analytically finds a result that fails to interpret. To solve this exercise, a young student who finished his secondary school recently, interpreted that *h* was the abscissa of the vertex, then he replaced -*h* by ½(b/a) and replaced *k* with *c*, then to pose equality: $a\left(x+\frac{b}{2a}\right)^2 + c = ax^2 + bx + c$ by which aimed to compare symbolically two different quadratic function expressions. When handling expressions he received $\frac{b^2}{4a} = 0$ and failed to interpret it. This student worked with a particular case, without realizing it, where the ordinate of the vertex and the ordinate at the origin are both *c*. Even without obtaining a general conclusion, he could be concluded that in this case $b=0$ and the expression is then corresponding to a particular family of quadratic functions, but he did not do so. The student acknowledged that the found result was not expected.

Syntactic aspect is prevalent here, it is shown in two items: student expresses in an algebraic register the abscissa of the vertex and replaces this in the original formula and he also replaces the parameter *k* by *c*, assuming that the independent term in both expressions must to coincide, the other item is the matching of the two symbolic expressions: $a\left(x+\frac{b}{2a}\right)^2 + c = ax^2 + bx + c$. Student replaces the literals *h* and *k* by expressions with *b* and *c* in order to arrive to the same literal expressions. In relation to the semantic aspect, faced with an expression that did not expects: $\frac{b^2}{4a} = 0$, he does not get giving a new meaning to the coefficients. In relation to the pragmatic aspect, in this case the student acknowledges the parameters that determine the vertex of the function and decides to write it with a symbolic expression that describes each of its coordinates in terms of the coefficients of the polynomial expression in order to obtain the same literal expressions.

The Determination of the Value of Truth of a Composed Proposition

We analyze the following statement where it appears a conjunction:

> *Decide if the following statement is true or false and justify the answer:*
> *The expression of the polynomial function* $f: R \to R / f(x) = 2x^3 - 2x^2 + 4x - 5$ *is divisible by the formula of* $g: R \to R / g(x) = x - 1$ *and by the formula of* $h: R \to R / h(x) = x + 2$

In relation to the pragmatic aspect, to resolve this activity some students (50%) used the Ruffini's rule and others the Remainder Theorem to determine if *f* is divisible by *g*. Who used the Remainder Theorem recognized that in order to decide on the divisibility of polynomial only knowing the rest of the division is required. In this case, the semantic context is more concise than carrying out the division and obtained more information than necessary, although both procedures used were correct. In both cases, students concluded that *f* is not divisible by *g*. Few students (16%) realized that since g did not divide *f* the statement was false and was not necessary to examine whether *f* was divisible by *h*. The most of students (84%) analyzed if *h* divided *f* with the same method used before, *h* does not divide to *f*. They concluded that the statement was false too. From a pragmatic point of view, students analyzed the validity of each simple proposition, but we note confusion about how these partial truth values determine the value of truth of composed proposition. We are left with the doubt of what would have been the response of these students in another statement that *f* would be divisible by *h*.

Immediate Inferences

Students were asked to consider the following statement:

> *Find the whole solution of the following inequality:* $f(x) > g(x)$ *where* $f: R \to R / f(x) = x^3 + x$, $g: R \to R / g(x) = 4x - 2$ *and* $f(1) = g(1)$

In class, the majority of students (75%) starting from the data $f(1)=g(1)$ they inferred that $x=1$ is the abscissa of a intersection point between the two graphs of functions. Not made use of the graphic register to view it. Other few students (25%) inferred that $x=1$ is a solution of $x^3+x=4x-2$ and were able to use the value $x=1$ for solving the equivalent equation $x^3-3x+2=0$ by factoring the polynomial and solving $(x-1)(x^2+x-2)=0$. Then they obtained all the abscissas of the points of intersection, but they not continued with the resolution of the inequality $f(x)>g(x)$.

In this case, students made an immediate inference by identifying a point of intersection of the graphs of two functions with the solution of the equation resulting by matching two function formulas. This information is extracted from the semantic interpretation of a syntactic expression corresponding to algebraic register: $f(1)=g(1)$, identifying that $x=1$ has the same image in both functions. However no student continued the resolution of the inequality, even through the numerical or graphical exploration, so we cannot give information on the pragmatic aspect of the task. At the same time, we observed that difficulties arose to express the information obtained from the inference in an algebraic register. The majority of the students failed to symbolically express the question corresponding to the search for the coordinates of the intersection of the graphs of the functions by means of an equation, which shows a limitation in syntactic and semantic aspects.

The Analytic Deduction

As was said above, in the analytic deduction symbolic manipulation is the engine that would make it possible to draw new conclusions. Mathematical symbols take an essential role in this logical form. The activity we exhibit below, of proportional function, shows the student's different responses on the linear properties, when the variables are extracted from a context, in a numerical way, and when they are out of context, in a symbolical way.

A Case with Linear Function

The first context introduced for studying linearity, is a problem of a conversion of currencies, with numerical values, problem in which there are slogans that invite you to infer such properties from reflecting on the accounts carried out in individual cases. Students addressed the problem alone with the dynamics of teamwork. They shared their approaches and, through the formula for conversion of currencies, in the context of the problem, the properties of linearity were concluded. They said that to convert a sum of quantities is the same that to convert each quantity and then to sum them and that for the double or triple of a quantity, the converted amount also increases to twice or three times respectively.

Then arose the following task aimed at an analytic deduction:

For the correspondence: $f{:}R \rightarrow R/\ f(x)=mx$,

a) *Write in symbols the corresponding by f of the variable x_1.*
b) *What is analytical development that allows you to verify that: "the corresponding of the sum of two values is the sum of the corresponding each one of them."? In the problem of currencies' conversion, find the calculations in which you work with this property.*
c) *What is the analytical development that allows you to verify that: "the corresponding of the product of a number and a variable is the product of the number and the corresponding by f of the variable."? In the problem of currencies' conversion, find the calculations in which you work with this property.*

Only four students solve a). For item b), from the students observed (33), only two managed to symbolically write. One wrote $f(x_1+x_2) = f(x_1)+ f(x_2)$ (expression 1) translating only the given statement and as not related it with the given formula for *f*, did not continue the deduction. Another wrote $f(x_1+x_2) = m(x_1+x_2)$ (expression 2) but did not continue with the deduction. When the expressions (1) and (2) were shared in class, they observed that by means of distributive property the second expression could be transformed in the first. It was easily identified where this property was used to solve the problem of currency exchange. The second condition $f(kx)=kf(x)$ was deduced without difficulty, by analogy with the previous case.

First we notice the difficulty of most of the students at the syntactic level to not be able to translate symbols expressed colloquially. For those two students who were able to do a translation, this was incomplete, only remained at a syntactic level and missed a semantic significance that would allow them to understand that in this theoretical context it was possible to apply the distributive property and thus verify the request. They needed a teacher orientation to this relation between the semantic and the syntax. At the pragmatic level, they used not even the resource of replacing by numbers the variables x or the parameter m to identify the involved mathematical property. Neither, they used the immediate previous problem where this issue had worked. In the second case it appeared the logical procedure of the analogy, to aid the deduction that was easily and properly obtained by all students in both planes: the syntactic and the semantic.

Deductive Chain

As we said, we analyze a student written protocol. We display an example in which we can notice a deductive chain, although the student not explicit conditions and intermediate conclusions in that string. We believe that the realization of graphics and the use of arrows connecting is a display mode that allows us to keep track of your reasoning.

A Case with Polynomial Function

In the following figure, we show the statement presented to the class and the response of a student:

Consider the polynomial function $f:\Re \to \Re$, $f(x) = 2(x^3 - 9x)(x^2 + 2x + c)$.
Find the value of c in order to $C^+ = (-3,-1) \cup (-1,0) \cup (3,+\infty)$
Student response:

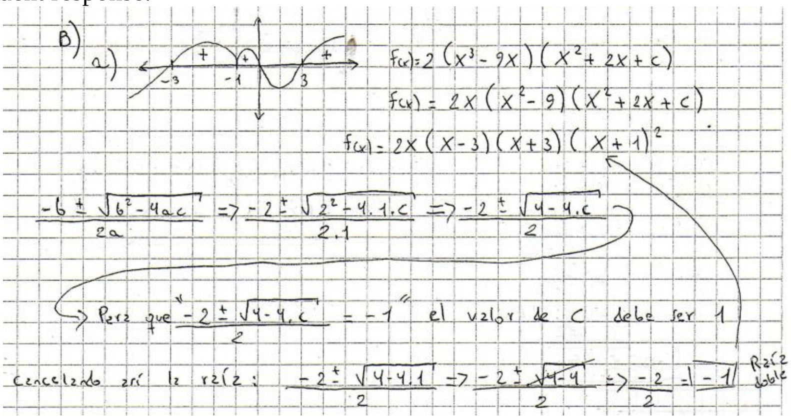

Figure 1. Resolution of a student. Example of deductive chain.

This 22 years old student was preparing to be admitted to industrial engineering. He makes a graph that represent polynomial function that met the condition of positivity. Besides the roots that obtain by algebraic manipulation, according to data and made graphic, it infers that it *x=-1* is root. Then he manipulates the polynomial expression to find the factor which "would impose him" the condition that -1 is root. He uses the quadratic equation solver formula to match the intended result *x=-1* in order to obtain the parameter. Although it is not explicit, the fact that the student verifies that the obtained value for c makes that x=-1 is a root of multiplicity two, makes us think that he understands the relationship between the condition of the multiplicity and the particularity of the given set of positivity.

In this case we see that he is permanently established a link between the syntactic, semantic and pragmatic aspects to solve the problem, although it is difficult to establish the limits of every aspect. From the syntactic point of view, we see different polynomial expressions and the formula for obtaining roots from a quadratic equation. We also consider the design of the Cartesian graph as part of the syntactic aspect, depending this of a semantic interpretation of the positivity of a function given in symbolic register of sets. In relation to the semantic, the student works on the polynomial expression increasing the number of factors to be able to give a new meaning the graphical information in terms of roots of this expression. Then he reinterprets that in *x=-1* the polynomial has a double root and he is able to use this information in a algebraic register to condition the coefficients of the quadratic formula, which would realize the dialog between the syntactic and semantic aspects. From the pragmatic point of view, we consider that the succession of steps: the realization of the graph, factoring, the use of quadratic formula, to matching this formula to -1 and the indication that the parameter *c* must be 1, allows the student "close a circuit" consistent with the interpreted information through the graph. Unfortunately, the conclusion does not allow us assuring the way of he obtains the *c*' value, if it is the result of algebraic manipulation or numerical exploration from the proposed equation.

The Refutation

After studying the exponential function and its characteristics, the following activity arose:

> *Decide if the following statement is true or false. Justify the choice. "The set image of the function f:R→R/* $f(x) = \left(\frac{1}{3}\right)^{x-2} - 3$ *is the interval* $[-3;+\infty)$ *".*

Student response*:* False. I tried replacing *y* by -3, since -3 is included in the image set $[-3;+\infty)$.

$$-3=\left(\frac{1}{3}\right)^{x-2}-3\Rightarrow 0=\left(\frac{1}{3}\right)^{x-2}$$

$$\log_{\frac{1}{3}} 0=\log_{\frac{1}{3}}\left(\frac{1}{3}\right)^{x-2}$$

No power of 1/3 resulting in “0”.

Figure 2. Example of refutation.

Figure 2 shows a case of reductio ad absurdum. This student (19 years old), had recently finished secondary school. He decides that the proposition was false through the recognition of a contradiction. To provoke it, he distinguishes an element that determines the falsity of the statement: the -3. Assuming the data about the image set, the -3 should satisfy the definition of image point, then it should exist a real value so that the proposed exponential equation is satisfied. Using essential features of the exponential, the equation was unsolvable. This type of deductions is very rare among the students. The most common approach is to draw the graph of the function and to determine the image set by observation of the graph. Then they compare, by means of graphic tools, the obtained with the set proposed in the task to determine that the statement was false. Those procedures illustrate the pragmatic aspect: on the one hand a procedure, based on observation, that compares sets and the decision is that the sets are not equal. On the other hand, the presented student shows an image candidate, belonging to the proposed interval, and uses the definition of image by a function leading to an equation. The choice of the element is appropriate to lead to the absurd. The registers manipulated in one and another case are not the same, in the first case it is symbolic and in the second it is graphic. However, in the second case, the arguments used are relevant and provide assurances for a community of production (could be the classroom) that accepts the graphic register as reliable.

For the response analyzed above, the semantic interpretation is that the image of the function proposed in the statement of the problem must satisfy an equation. The algebraic expressions of the manipulation combines the syntactic aspect, using correct symbols, and the semantic one in the use of exponential and logarithm. Finally, when the student solves the equation he finds that it has no solution but he does not concludes explicitly that the element in question,-3, does not belong to the image.

Conclusions

The intention of our work is not to point out students’ mistakes or to measure how incomplete the responses are under the logical point of view. On the contrary, we want to show that even in responses, that are apparently not reached, there is logical complexity. Combining syntactic, semantic and pragmatic aspects for analyzing mathematical production give visibility to logical notions present in the students’ mathematical activity that sometimes is underestimated in the instruction or assessments.

After studying the logical constructions in productions of students under this combination, we notice that: To identify membership in a class of a family of objects, we see a variety of pragmatic resources: to put a equality for obtaining conclusions on parameters, replace the parameters by numbers, and a case where the parameters are replaced by known literal formulas in order to make coincide literal coefficients of two different expressions. For the determination of the value of truth of a composed proposition, the more habitual observed response is that students make the calculus or the applications of properties in order to determine the truth value of each partial proposition although it is not necessary. In the semantic aspect, this allows to establish as hypothesis that exhaustiveness in the validity evaluation of each part of the complete proposition.

With respect to each of the logical construction of reasoning, we see that the forms of refutations, as reductio ad absurdum, recognition of the contradiction and the counterexample, are rare in students while in the class both the teacher and the learning activities insist with them. This could be explained by the cognitive complexity that required generate and use own examples or particular cases built appropriately from concepts, definitions and properties management. This complexity, linked with the semantic aspect, conditions the pragmatic aspect since it reduces the ways of dealing with a problematic situation. In terms of the analytical deductions, few students manage to make those deriving mathematical properties. Among the cases analyzed, we see that the first of them, where requesting an analytic deduction that led to a property (the linearity of the sum of the proportional functions), was the most difficult, even though the properties and manipulation involved are simple and was worked and understood in a numeric context and with a familiar situation. This suggests that the nature of the mathematical task influences the generation of formal constructions and, possibly, students are more used to the analytical deductions when the requested task is to get some kind of object: a numeric value, a function, a polynomial, etc. than when they have to deal with a property.

After studying the logical constructions in students' productions for its analysis pursuant to syntactic, semantic and pragmatic aspects, we notice that: Regarding the syntactic aspect, we observe in the analyzed productions that deductive chains lack the linguistic connectors that translate into logical connectors. Paragraphs are segmented without internal coherence, so that even if the conclusion is correct is not clear how they achieve them. On the positive side, we see that in many cases, the information is presented in different representation registers that in which the initial data is given. This rewrite was not arbitrary or mechanical but that it was properly selected with the intention of outlining information, symbolize it, manipulate it, etc. to later infer new information that would address the problem. In some cases, as in the exercise on proportionality, we observe that the symbolic language used so naturally in math classes by teachers in general, turned out rather hindering in the analytical deductions because of the confusion between the sign and its meaning.

In relation to the semantic aspect, we notice that a resource is to put equality at the beginning of the analysis or the resolution. By putting equality, objects are "encapsulated" and it locks the access to the quantifiers that eventually appear in the definition. Also, by this procedure the analysis is complicated because in every step is necessary to have a clear knowledge about the elements manipulated in the equality. In this regard, we notice that, in order to perform algebraic manipulations to reach some results, students use a same letter with different meanings in a same symbolic expression.

As we have seen, the pragmatic aspect refers to personal modes of action in relation to the formal construction. These actions are analyzed as ways of doing and not only in the sense of doing the mathematically correct. The pragmatic aspect is not efficient, without a previous semantic framing that demonstrate some understanding of the objects with which they work, definitions or properties. We have applied for the analysis the Durand-Guerrier's (2004, 2005, 2008) hypothesis that stresses that it is necessary to take into account the syntactic, semantic and pragmatic aspects of the logical reasoning. We have enlarged the class of analyzed objects to the formal constructions, not only for reasoning, but also for concepts and judgements and we find that this hypothesis is a good criterion. We see, through the study of cases, as when the student manages to adequately link the semantic and syntactic, their formal constructions are presented in a way that is more complete and appropriate for the proposed problem than those that is driven by only one of the previously mentioned aspects. Analyses show that if it has been previously achieved some fairly correct bonding between the semantic and the syntax, the shares corresponding to the pragmatic aspect have more efficiency's value, because they approximate the student to some conclusion. This reinforces the opinion of Easdown (2006) for whom the development and exploitation of the links between the syntax of symbolic representation and the underlying semantics is perhaps the most important function of a teaching and learning of mathematics successful.

This work provides an identification and analysis of formal constructions from mathematic tasks in which the requirement of the use of logic was not explicit but present. We believe that the analysis of the underlying logic in math customary tasks, at the first courses of the university, contributes to mathematics education at this level. From the didactical point of view we conclude that it is essential to insist on the use of the oral and written language, not only on the appropriate use of symbols. Quotidian language is the means for acceding to meaning. Analyzed productions also make us to think that it is necessary to design tasks that pose equalities of expressions have different consequences, not only to solve an equation, for example: to find parameters, to put conditions, to find curves' intersection, etc. The resolutions of those tasks must be accompanied with reflection on those uses. Teaching must also put in evidence different pragmatic actions in face to a same task and analyze their scopes.

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