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Abstract

The purpose of this study was to examine patterns of mathematical practices evident during children's interactions with touchscreen mathematics virtual manipulatives. Researchers analyzed 33 Kindergarten children's interactions during activities involving apps featuring mathematical content of early number sense or quantity in base ten, recorded during one-to-one task-based interviews. Iterative analysis involved applying learning progression rubrics to video data, using hierarchical clustering to visualize the progressions via heatmaps with dendrograms, and returning to video data to investigate emergent patterns. Results indicated that overall, children's mathematical practices aligned with research on development of mathematical understandings, but that individual children's mathematical practices changed little within a given activity. The study extends existing research by shedding light on children's interactions with touchscreen mathematics virtual manipulatives and supports the use of these analysis and visualization techniques.

Introduction

Touchscreen technology is becoming ubiquitous in classroom environments, with various mathematics apps on mobile devices gaining popularity. Often, mathematics apps include virtual manipulatives, which are "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (Moyer-Packenham & Bolyard, 2016, p. 13). Although research on virtual manipulatives suggests that they can be effective tools for learning mathematics (e.g., Moyer-Packenham & Westenskow, 2016; Satsangi & Bouck, 2014) less research exists focusing on children's mathematical learning while interacting with touchscreen virtual manipulatives (e.g., Paek, Hoffman, & Black, 2016; Tucker, 2016). Additionally, studies of children's mathematical learning often involve many individual tasks, akin to the numerous tasks a child might complete in the classroom environment, and can generate rich data sets (e.g., Holgersson et al., 2016). Visualizations of these rich data sets often aid interpretation, and different visualizations serve different purposes. Heatmaps can draw attention to occurrences of certain values in the data (e.g., individual scores), while hierarchical clustering arranges data to ease identification of groupings based on the data. Therefore, the purpose of this exploratory study was to examine patterns of mathematical practices evident as children interacted with touchscreen mathematics virtual manipulatives, using heatmaps and dendrograms as visualizations during the analysis process.

Interacting with Technology to Learn Mathematics

Research suggests that children can learn mathematics while interacting with technology. This section synthesizes relevant theoretical foundations for learning mathematics, particularly while using technology, and the mathematics involved in this study. Children can learn mathematics by engaging in mathematical practices involving physically embodied interactions with representations. Mathematical practices are what mathematicians and mathematics users do, including justifying mathematical claims, demonstrating reasoning, and interpreting and using symbolic notations to apply their mathematical knowledge (Ball, 2003). Representations are internal (i.e., mental) configurations or external (i.e., physically embodied) configurations of mathematics (Goldin & Kaput, 1996), and internalization and externalization of representations occurs during mathematical practices. Theorists use embodied cognition to describe mathematical learning through physical interactions with external representations (i.e., what mathematics users physically do to learn math). When children use a tool as part of their mathematical practices, children perceive and interact with representations in such a way that integrates thought and action (Nemirovsky, Kelton, & Rhodehamel, 2013). Applying this view of embodied cognition, physical engagement in mathematical practices is equivalent to mathematical thinking, and changes in embodied mathematical practices is equivalent to mathematical learning.

Technology can support learning of many mathematics concepts, from basic number sense (e.g., Clements & Sarama, 2007) to introductory Calculus (e.g., Patenaude, 2013). Studies using various methodologies, including design-based research (e.g., Holgersson et al., 2016), experimental repeated measures (e.g., Riconscente, 2013), and meta-analyses (e.g., Moyer-Packenham & Westenskow, 2016) indicate that using virtual manipulatives can have positive effects on mathematics learning. However, simply using such technology does not guarantee improvements in children's achievement (e.g., Burns & Hamm, 2011; Moyer-Packenham, Shumway, et al., 2015). Thus, many researchers have examined children's interactions with technology, such as mathematics apps, finding that many complex elements involved in children's interactions can contribute to the development of mathematical practices. These elements can include app characteristics (e.g., Larkin, 2015), interaction modalities (e.g., Paek et al., 2016; Sinclair & Pimm, 2015), affordance-ability relationships (Tucker, Moyer-Packenham, Westenskow, & Jordan, 2016), interactions with peers (e.g., Baccaglini-Frank & Maracci, 2015; Sinclair, Chorney, & Rodney, 2015), and children's characteristics (e.g., Desoete, Praet, Velde, Craene, & Hantson, 2016; Moyer-Packenham & Suh, 2012). Emerging research indicates that children may not spontaneously reflect on their strategies (i.e., mathematical practices), even if their strategies change (Baccaglini-Frank & Maracci, 2015). Studies of the interactions and strategies may involve detailed, theoretically grounded qualitative descriptions (e.g., Sinclair et al., 2015; Sinclair & Pimm, 2015) and categorization of patterns (e.g., Baccaglini-Frank & Maracci, 2015; Tucker et al., 2016). Although these studies provide vivid depictions of mathematical practices and may offer frameworks for interpretation, they are not intended to offer data visualizations that facilitate a wider, pictorial view of the larger data set. Few relevant studies include data visualizations based on quantitized data (e.g., Tucker, 2015) or quantitative data (e.g., Moyer-Packenham, Tucker, Westenskow, & Symanzik, 2015). Hence, there are a myriad of opportunities and methods to examine mathematical practices that contribute to the outcomes associated with children's mathematical learning while using technology.

The mathematical content areas of focus in this study were early number sense and quantity in base ten, specifically targeted for children ages five to six years old. Early number sense involves a variety of constructs, including subitizing, part-whole relations, and counting. Subitizing involves recognizing how many objects are in small quantities without counting (e.g., Butterworth, 2005). Part-whole relations include recognizing quantity and potential composition and decomposition (Resnick, Bill, Lesgold, & Leer, 1991). Counting is distinct from subitizing and part-whole relations, and involves components such as cardinality and one-to-one correspondence (Sarama & Clements, 2009). However, children may use both counting and subitizing to identify quantities as part of early number sense. Understanding quantity in base ten involves coordinating concrete and symbolic representations of place value in the base ten system, including grouped objects (e.g., "ten" block), numerals, and number words (Fuson et al., 1997). These are developmentally appropriate foundational concepts for number sense and arithmetic (Sarama & Clements, 2009). Specifically, most children aged five to six years old should be able to: a) recognize groups of up to ten objects in structured arrangements without counting all individual objects, and b) begin to use groups of tens to recognize quantities greater than 10. However, most children aged five to six years old would not be expected to fluently use base ten blocks as tens and ones for counting purposes in their mathematical practices. Furthermore, these understandings build through a mix of development (e.g., elements of subitizing) and experience (e.g., part-whole relations). These mathematical practices involve an array of interrelated mathematical constructs (e.g., finger gnosis: representing a quantity with one's fingers), but the focus of this study is on the relatively general categories as delineated. Although research on children's interactions with technology to learn mathematics is growing, research specifically investigating young children's learning related to foundational elements of number sense and arithmetic while using touchscreen virtual manipulatives is in its infancy (e.g., Holgersson et al., 2016; Ladel & Kortenkamp, 2016). Given the plethora of technologies and their increasing popularity in the mathematics classroom (e.g., Calder, 2015), more research is required to explore this complex field.

Heatmaps and Dendrograms

Heatmaps and dendrograms are often used to visualize data. *Heatmaps* are a widely used, easy-to-interpret, graphical tool to summarize data, observe similarities and differences among several statistical variables (the columns in a heatmap), and compare and order observations from numerous participants of a study (the rows in a heatmap). Heatmaps and related graphs originally were introduced in the 19th century (Wilkinson & Friendly, 2009) and have become popular in genetics and biostatistics (e.g., Eisen, Spellman, Brown, & Botstein, 1998).

Heatmaps have been occasionally used to display data in education research, such as to display the temporal development of students' behaviors in learning environments (Kinnebrew, Segedy, & Biswas, 2014). The main idea underlying a heatmap is to split the data into several uniform intervals and then assign a different color to display each interval. Divergent color schemes (with distinctive colors for the intervals) or sequential color schemes (with interval shades changing from light to dark) are widely used. The rows and columns of a heatmap can be arranged in several ways, e.g., in sequential order of the participants' ID numbers or in alphabetical order of the variables – or they can be rearranged in such a way that rows (or columns) that are more similar to each other are placed next to each other in the graphical representation.

Dendrograms are tree-based graphical representations of the results of a hierarchical clustering algorithm. Terminal nodes in a dendrogram represent individual observations and inner non-terminal nodes represent groups of at least two observations. The height of each inner node is an indicator of how similar (or dissimilar) the observations are in the branches that lead towards the terminal nodes. When multiple individual observations are connected at height zero (i.e., directly at the terminal node height), this implies that all numerical values are exactly the same for these observations (see section 14.3.12 in Hastie, Tibshirani, & Friedman, 2001). In education research, combinations of heatmaps and dendrograms have been used occasionally, such as to display the results from a hierarchical cluster analysis of a longitudinal K-12 grading history data set (Bowers, 2010) and to examine second graders' use of mathematics virtual manipulative apps (Moyer-Packenham, Tucker, et al., 2015). To create dendrograms, the similarity of sorted rows and columns is determined via *hierarchical clustering* (see Section 14.3.12 in Hastie, Tibshirani & Friedman, 2001). The resulting dendrograms are displayed on the left or right of the heatmap for the rows, and, when sorted, to the top or bottom of the heatmap for the columns. Combinations of heatmaps and dendrograms can aid analysis of large, rich data sets, such as those that occur when children complete many individual tasks, as in this study. The visual combination of the colors from the heatmaps and dendrograms from the hierarchical clustering facilitates pattern recognition that may be more difficult in other methods of analysis.

The pairing of heatmaps and dendrograms allows for a much richer understanding of the resulting visual than either analysis method alone. Figure 1 (left) displays an example of a heatmap without the hierarchical clustering applied. While the cluster of Child 47 – Child 50, labeled along the right, is easy to distinguish, it is much more difficult to recognize that Child 41, Child 39, Child 52, Child 68, and Child 36 also belong in this cluster. In addition, one might overlook Child 43 as a candidate for this cluster due to the larger amount of tasks completed and the third task, which is at a higher learning progression score. In contrast, Figure 1 (center) displays the associated dendrogram sorting the children into the different clusters. Without the accompanying heatmap, it is impossible to tell what performance these clusters represent. Even coupled with an average performance measure, changes in performance (e.g., Child 43) would not be observable. Consequently, Figure 1 (right) pairs the heatmap with the dendrogram to provide an efficient, comprehensive view of the children's performance on this activity.

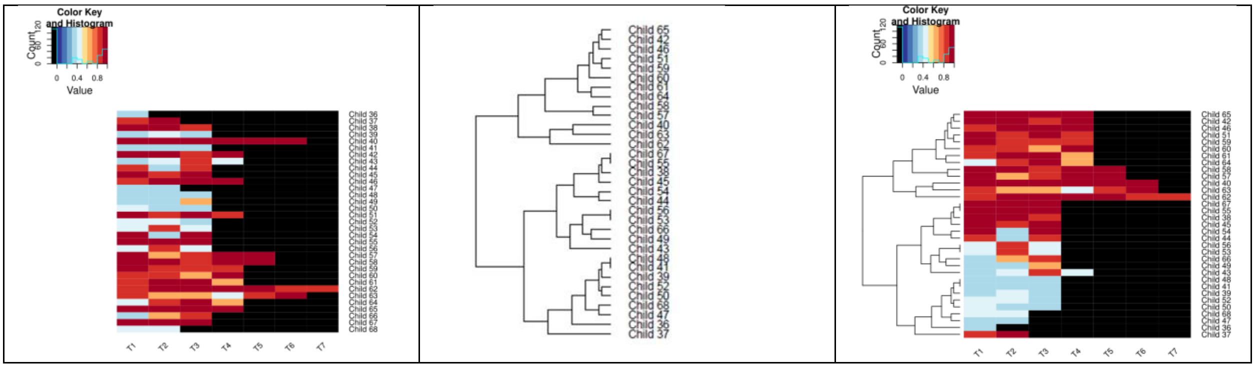

Figure 1. Example of a heatmap without dendrogram or sorting from hierarchical clustering (left), dendrogram from hierarchical clustering (center), heatmap with sorting from hierarchical clustering and associated dendrogram (right)

Methods

This mixed methods exploratory study (Creswell & Plano-Clark, 2011) examined children's mathematical practices as they interacted with multiple touchscreen mathematics virtual manipulative apps. In particular, this study examined a subset of children from a larger study that involved additional grade levels and apps (e.g., Moyer-Packenham, Shumway, et al., 2015) while integrating video coding, heatmaps, and hierarchical clustering in the analyses. The research question that guided this study was: What patterns of mathematical practices are evident as kindergarten children interact with touchscreen mathematics virtual manipulative apps?

Participants

The participants in this study were 33 children enrolled in Kindergarten, ages 5 and 6. To recruit the participants, researchers supplied local elementary schools in a county in the Intermountain West of the United States with informational brochures and letters to distribute to parents. Children's parents reported that most children were Caucasian (91%) and nearly half (48%) indicated low socio-economic status (i.e., free/reduced lunch at school). Children's parents reported that most of their homes had touchscreen devices (88%) and that most children used a touchscreen device at least once per week (88%). Researchers assigned each participant a code from #36-68 for confidentiality and data analysis.

Procedures

The study began with app selection, interview protocol design, and the creation of data collection procedures. Researchers chose the iPad as the touchscreen device because of its portability, usability, and array of available apps. Researchers piloted the apps and protocols with children in local schools to refine the protocols and procedures. The apps and tasks were chosen because they required interactions with relevant, developmentally appropriate mathematics content (e.g., Sarama & Clements, 2009) via virtual manipulatives that featured affordances linked to positive learning outcomes (e.g., Moyer-Packenham & Westenskow, 2016). (For detailed discussion of links between affordances and learning outcomes, see Moyer-Packenham et al., 2016.) The apps and activities within the apps were: Hungry Guppy: 4-dot and 5-dot Fish, Fingu: Levels 1 and 2, Friends of Ten: How Many More (10 Frame), and Montessori Numbers: Math Activities for Kids: Quantity 10-99; 1 to 20: 11-20; and Numerals. In this context, apps refer to standalone applications, activities refer to distinct parts of the app, and tasks refer to app- or researcher-generated problems for the child to complete. Descriptions and screenshots of the activities and tasks appear in the corresponding Results sections.

Each child participated in a one-to-one task-based interview in a room featuring a two-way mirror connected to an observation room, located in a research center on a university campus. Task-based interviews allow children to demonstrate mathematical understandings and practices in response to prompts (Goldin, 2000). In this context, children interacted with technology to demonstrate mathematical understandings and practices in response to tasks presented by the technology and a researcher. While one researcher conducted a child's interview, a second researcher recorded observations from the observation room. Each interview was comprised of two parts: Quantity in Base Ten (QBT), then Early Number Sense (ENS). Every child interacted with the same activities, but the order of learning activities within the sequence varied to control for potential order effects. During the first part of the interview (QBT), children completed tasks using activities in the Montessori Numbers: Math Activities for Kids app. Children began with the pre-assessment using the Quantity 10-99 activity, followed by learning tasks using the 1 to 20: 11-20 and Numerals activities, and then the post-assessment using the Quantity 10-99 activity. During the second part of the interview (ENS), children completed pre-assessment tasks using the 10 Frame activity, followed by learning tasks using Hungry Guppy and Fingu apps, and then the post-assessment using the 10 Frame activity. Interviews took approximately 30 minutes per child.

Data Sources

Data sources for this study centered on task-based interviews with children. Each interview was recorded using two camera angles: a wall-mounted camera to provide an overhead view and a wearable GoPro camera to provide a "child's-eye" view. Researchers in the observation room noted occurrences that the cameras might miss, including affective responses, interviewer actions, and unique occurrences (e.g., bathroom break). The recordings and observation notes provided multiple perspectives for analysis (Lesh & Lehrer, 2000).

Data Analysis

Data analysis involved quantitatively and qualitatively analyzing video data, generating heatmaps and dendrograms, and integrating the analyses. To analyze children's video-recorded mathematical practices during their interactions with the touchscreen mathematics virtual manipulatives, researchers developed learning progression rubrics for each task. Learning progressions are "descriptions of successively more sophisticated ways of reasoning within a content domain based on research syntheses and conceptual analyses" (Smith, Wiser, Anderson, & Krajcik, 2006, p. 1). Researchers tailored the learning progressions to the content (e.g., early number sense) as it manifested in the specific activities (e.g., Fingu, level 1 and 2) involved in the study. To develop the learning progression rubrics, researchers viewed videos and identified initial codes for children's mathematical practices, such as strategies to complete tasks, applying research in mathematics learning (e.g., Sarama & Clements, 2009) to this context. Researchers then iteratively created and refined a learning progression coding typology for each activity. The final coding rubrics emerged from this process. Finally, researchers applied the codes to all videos. To ensure internal validity, 20% of the videos were double-coded by two researchers. Researchers also noted occurrences outside of the learning progressions (e.g., waiting for animations, rushing, or favoring particular colors of objects).

Each finalized rubric represented the hypothesized learning progression and levels of reasoning for the content (i.e., components of early number sense or quantity in base ten) as children engaged in mathematical practices while interacting with the app. For example, there were six identified levels of quantity reasoning children could demonstrate while using the Montessori Numbers: Math Activities for Kids app, Numerals activity (see Table 1). These ranged from guessing without counting, which scored a 1, to a response that demonstrated understanding of counting on unitizing (e.g., 10, 20, 30, 31, 32, 33, 34), which scored a 6.

Table 1. Learning progression scoring rubric for montessori numbers: Math activities for kids: Numerals

Level	Mathematical Learning Progression Expectations; Growth in Base-10 Number Sense
1	Child guesses; no response
2	Pre-counting: child says number names but does not match to objects; treats tens and ones as equal units
3	Counting by ones: child displays confidence in cardinality with ones and identifies the correct numeral for the set; can count the entire set by ones; does not use groups of ten to count the set (Unitary multi-digit conception). Possible evidence: counts tens block by ones (1, 2, 3, …10) instead of immediately counting by tens (10, 20, 30, …)
4	One-to-one counting: child recognizes and counts tens and ones; matches each spoken number with one and only one object (10 or 1), but is unsure of cardinality with tens and ones (i.e., cannot tell how many)
5	Counting tens and ones separately: child correctly counts tens and ones and identifies correct numeral to match the set; does not show evidence of combining tens and ones (e.g., first, counts ones and identifies ones numeral, and then counts tens and identifies the tens numeral). (Decade-and-ones conception)
6	Counting on/unitizing: child counts on combining tens and ones (e.g., 10, 20, 30, 31, 32, 33, 34) (Separate-tens-and-ones conception)

The scores from these learning progression rubrics were used in a hierarchical clustering analysis to generate heatmaps paired with dendrograms to visualize the data. For these heatmaps, the observations of each variable were rescaled from 0 to 1. Then, ten intervals of width 0.1 were created. These ten intervals were mapped to a blue/red divergent color scheme. Darker blue colors represent values closer to 0 and darker red colors represent values closer to 1. Some activities involved varying numbers of tasks. For children with fewer completed attempts, the missing attempts are encoded with the value "-0.1" and are represented in black on the heatmaps. The colors and numerical intervals appear in a legend in the upper left corner of each heatmap. A histogram overlaid on the color representation shows how many observations occurred for each interval across the entire heatmap. The rows of the heatmaps (which represent participating children) are sorted based on the similarity of the observations for each child. In most cases, the columns of the heatmaps (which represent tasks within each app) are also sorted. Heatmaps where the columns are not sorted required preservation of the temporal order of the tasks.

Researchers then integrated the heatmap and hierarchical clustering analysis with the video analysis. This involved comparing patterns of mathematical practices, unique cases, and clusters found in the heatmaps and dendrograms with the ongoing iterative coding of the video data to determine in-depth explanations for these phenomena.

Results

The results are presented in two sections: first is the learning progression scores from the Quantity in Base Ten (QBT) Sequence and second is the learning progression scores from the Early Number Sense (ENS) Sequence. Each section describes the app, activities, and corresponding tasks followed by the results for all children with details for notable patterns and unique cases. The results include all children who attempted each set of tasks in a particular sequence; thus, there were 33 children who participated in the QBT Sequence and 30 children who participated in the ENS Sequence.

Quantity in Base Ten (QBT) Sequence

The QBT Sequence included tasks using three activities within the Montessori Numbers: Math Activities for Kids app: Quantity 10-99, 1 to 20: 11-20, and Numerals.

Quantity 10-99

In the QBT Sequence, the children began with the Quantity: 10-99 activity as a pre-assessment where they completed three quantity-building tasks (see Figure 2). Using the Base-10 Blocks, the children built the quantities 14, 31, and 50. To build these quantities, children used tens rods and units blocks. The children repeated these three tasks as a post-assessment after completing tasks using the two learning activities.

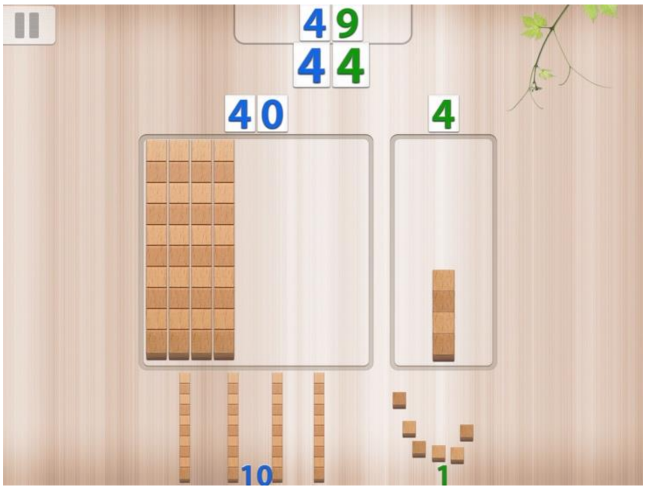

Figure 2. Screenshot of montessori numbers: Math activities for kids app: Quantity 10-99 activity

The heatmap in Figure 3 displays children's performance on the pre- and post-assessments. The tasks (e.g., PreT1:14) are grouped by similarity as calculated by hierarchical clustering and displayed using the associated dendrogram. Children are grouped by similarity of their performance.

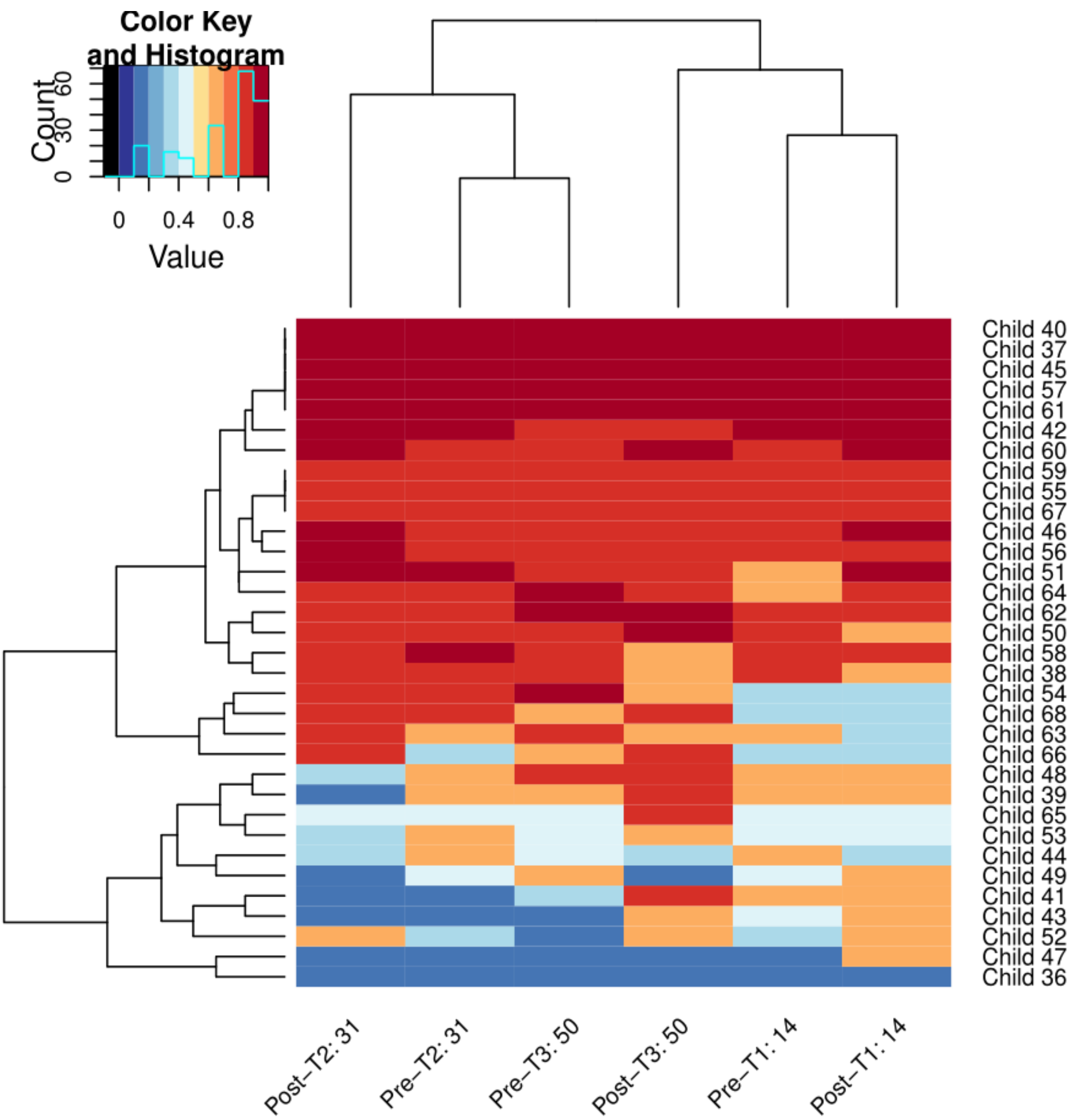

Figure 3. Heatmap and dendrograms of tasks for the pre- and post-assessments in the QBT sequence

There are four main clusters of children. The first cluster of children, Child 40 – Child 38, attained the highest three learning progression scores on all tasks in the pre- and post-assessments. Of the 18 children in this cluster, 13 children had scores that did not fall below the top two highest learning progression scores. The second cluster of children, Child 54 – Child 66, attained mixed learning progression scores ranging from 0.3 to 1.0. The third cluster of children, Child 48 – Child 52, received lower mixed learning progression scores. The fourth cluster, Child 47 and Child 36, received consistently low learning progression scores across all tasks, but one. Examining the heatmap by task, the post-assessment task to build 31 (Post-T2: 31) proved to be the most difficult with six children obtaining the lowest learning progression level.

Children who achieved higher learning progression scores, those in the first cluster, consistently correctly used both the tens and the ones to create the target number. This is in contrast with the rest of the children, who frequently interchanged the tens and ones for counting from 0-9 or used only ones to build numbers, disregarding the tens. Some children changed their mathematical practices from the pre- to post-assessment. For example, Child 41 and Child 65 changed from using only ones to build the target number of 50 (i.e., counting by ones as 1, 2, … 50) in the pre-assessment to using the tens to build the target number of 50 (i.e, counting by tens as 10, 20, … 50) in the post-assessment. However, while Child 52 and Child 43 changed their mathematical practices from never using tens to attempting to include tens on multiple tasks, no children changed from never using tens to consistently correctly using both tens and ones on multiple tasks.

1 to 20: 11-20

In the QBT Sequence, the children interacted with two learning activities between the pre- and post-assessment. The first learning activity, 1 to 20: 11-20, provided the children a scaffolded experience in building the numbers 11-20 using base-10 blocks. The activity first showed the child how to build the number using the base-10 blocks and then allowed the child to try. The interviewer prompted children to build 11, 15, and 20 (Tasks 1, 2, and 3, respectively).

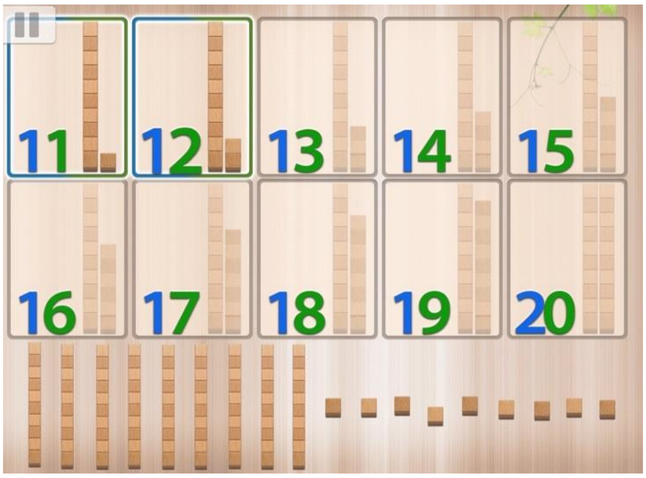

Figure 4. Screenshot of montessori numbers: Math activities for dids app: 1 to 20: 11-20 activity

This heatmap displays the children's performance on the first learning activity tasks (see Figure 5). The tasks are indicated along the bottom of the figure with T1: 11, T2: 15, and T3: 20. The tasks are grouped by the similarity using the dendrogram at the top of the figure while the children are grouped by the similarity of their performance using the dendrogram to the left of the figure.

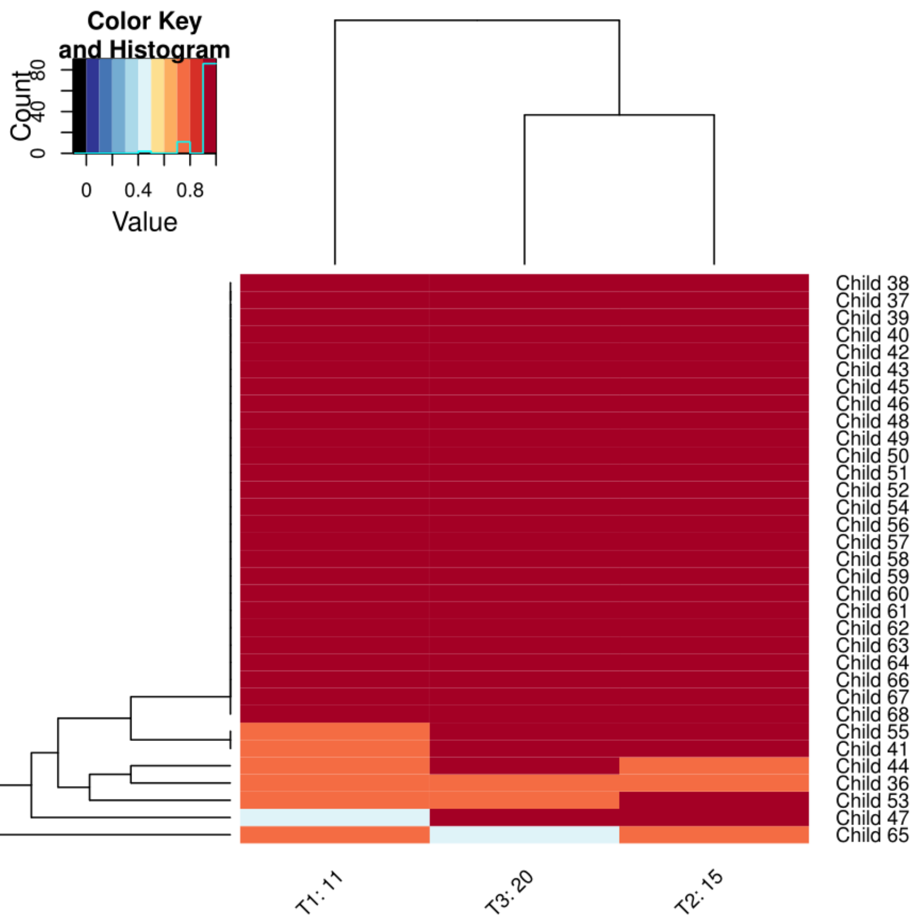

Figure 5. Heatmap and dendrograms of tasks for the 1 to 20: 11-20 learning activity in the QBT sequence

Twenty-six of the 33 children attained perfect scores of 1.0 on all three tasks of this learning activity, creating the first cluster of Child 38 – Child 68. The second cluster of children, Child 55 – Child 53, attained either perfect scores or scores of 0.7 on the three tasks. In the third cluster, two children, Child 47 and Child 65, attained scores lower than 0.7 on a single task each. Returning to the video data revealed two main differences in mathematical practices and affective responses between children who attained all perfect scores and those who did not: a) their use of the tens rod to represent ten, and b) patience with the activity structure. The children who attained all perfect scores efficiently used the tens rod while modeling the assigned quantities, waiting long enough for the animations to occur so they did not make mistakes. Of the children who did not attain all perfect scores, some did not immediately recognize that a tens rod was more efficient than ten ones for building the quantities, such as Child 41 and Child 55, who did not begin using the tens rod as ten until the second task, and Child 36, who only used ones. Child 65 did not attain perfect scores due to a lack of patience with the animations associated with the tasks presented, making mistakes while rushing through the tasks.

Numerals

The second learning activity in the QBT Sequence was Numerals (see Figure 6). In this learning activity, the base-10 blocks construction of a number is shown using tens rods and unit blocks. Children must then use the numerals to indicate the quantity shown in the construction, receiving sparkles as feedback for a correct response. The interviewer provided instructions by demonstrating how to count the blocks to determine what quantity is represented. Children could complete as many tasks as the interview time allowed for.

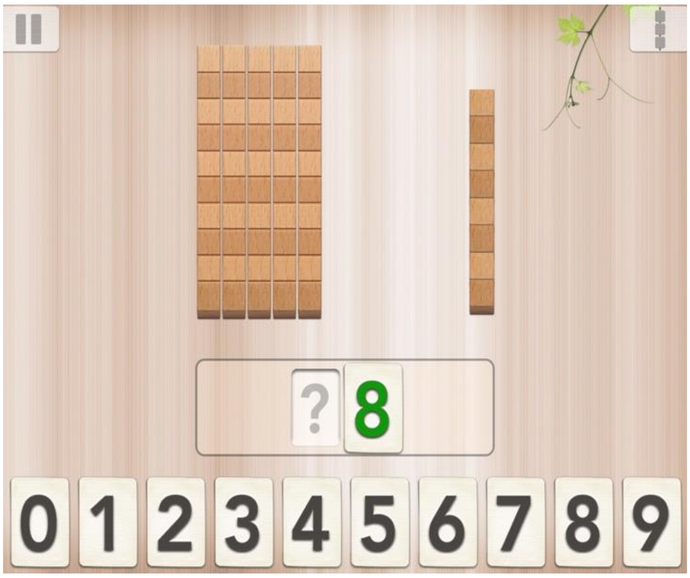

Figure 6. Screenshot of montessorai numbers: Math activities for kids app: Numerals activity

In this heatmap, the blocks in the blue-red spectrum represent the children’s learning progression scores on that task. The tasks labeled along the bottom indicate the number of tasks that the child completed. Black blocks indicate tasks that the child did not complete within the allotted interview time.

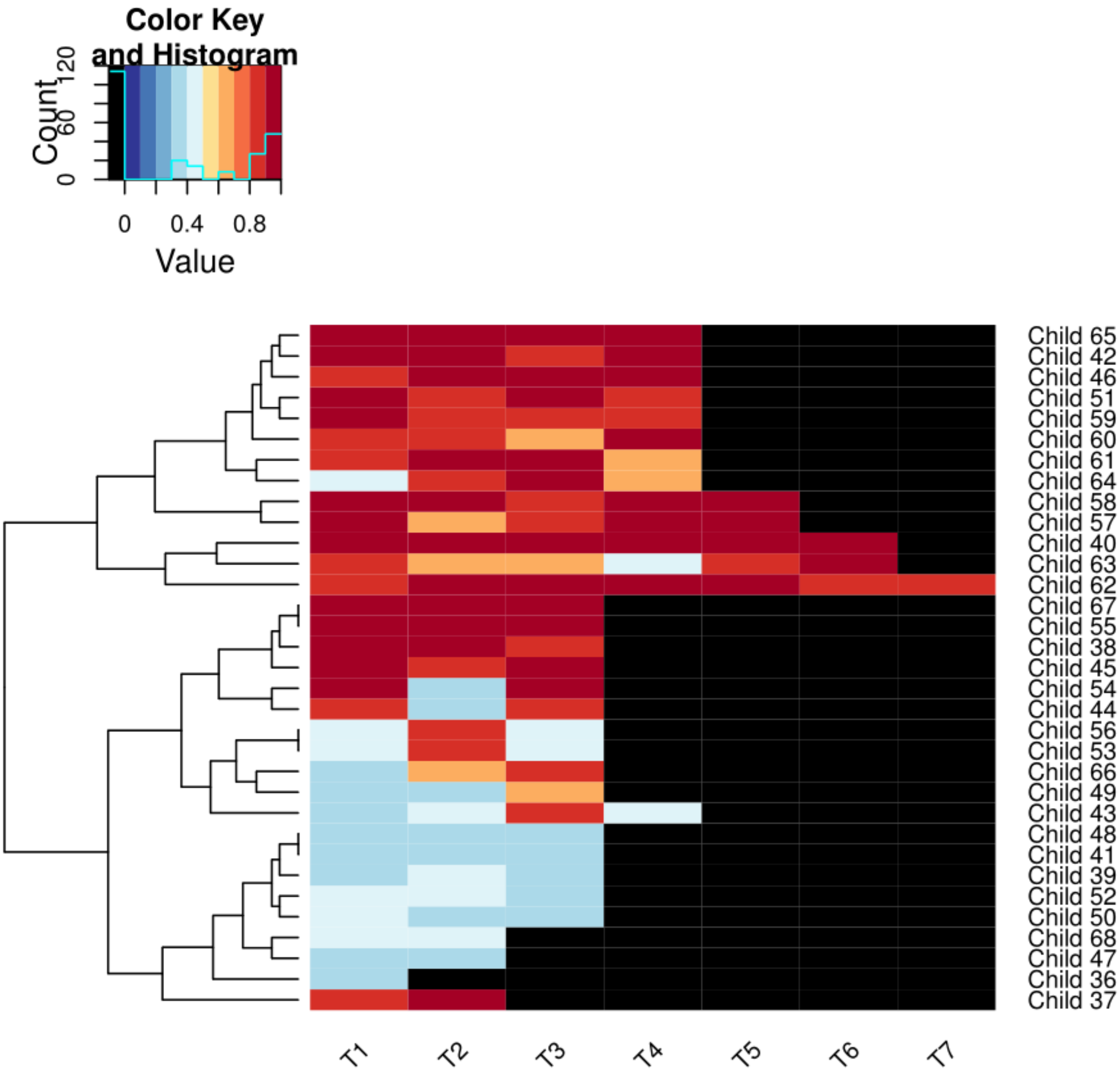

Figure 7. Heatmap and dendrogram of tasks for the numerals learning activity in the QBT sequence

In this learning activity, there are two primary clusters of children. The first cluster of children, Child 65 – Child 62, completed between four to seven tasks nearly all of which were rated with a learning progression score of 0.6 or higher. The second cluster of children, Child 67 – Child 37, completed between one and four tasks. This cluster is further broken down into two clusters: one of children who achieved higher learning progression scores and completed more tasks, Child 67 – Child 49, and a cluster of children who completed fewer tasks and/or attained lower learning progression scores, Child 48 – Child 37.

Child 40 and Child 63 from the first cluster provide an interesting comparison. While Child 40 exhibited a pattern of attaining the highest learning progression scores on all tasks, Child 63's performance was more variable, as indicated by the learning progression scores ranging from 0.4 to 1.0. Returning to the video showed that Child 40 appeared very confident in counting the tens rods as tens and unit blocks as ones, though on one task Child 40 miscounted and immediately self-corrected. In contrast, Child 63 first counted each block individually not recognizing that each tens rod was composed of ten unit blocks. When asked if there were other ways to count the blocks, Child 63 suggested counting by fives before progressing to counting by tens. When asked to identify the quantity shown by five tens rods and a single unit block, Child 63 counted the tens as "10, 20, 30, 40, 50" but then continued counting with the unit block as "60". This mathematical practice demonstrated the disconnect between Child 63's rote skip counting skills and the quantities the counting represented. Child 63 repeated this error in several tasks, but often self-corrected after realizing the initial response was incorrect when the app did not provide the sparkles feedback. On the final task, Child 63 recognized that the tens rods did not need to be counted individually, thus altering his mathematical practice to a more efficient one. The patterns observed in this heatmap were particularly useful in identifying the very different performances of these two children who both completed the same number of tasks and were clustered together in the dendrogram, leading to a deeper analysis of the mathematical practices and understandings these children showed.

Early Number Sense Sequence

The ENS sequence included tasks using three apps: Friends of Ten, Hungry Guppy, and Fingu.

Friends of Ten: How Many More (Ten Frame)

The Friends of Ten app included activities using virtual ten frames and served as both the pre- and post-assessments. The How Many More (Ten Frame) activity presented colored discs on a ten frame below a written prompt asking "how many more to" a given target number (see Figure 8). First, the researcher asked the child how many discs were present on the ten frame. After the child replied, the researcher asked "how many more to" the target number. The child then dragged colored discs onto the ten frame to reach the target number and indicated the quantity of discs added using numbers shown below the ten frame. The pre-assessment involved determining how many more to 4, 7, and 9, while the post-assessment involved determining how many more to 5, 7, and 9. For each task, the starting quantity, disc color on the ten frame, and disc arrangement varied.

Figure 8. Screenshot of friends of 10 app: How many more (ten frame) activity

Figure 9 shows a heatmap of the combined pre- and post-assessment results for the 30 children that completed the ENS sequence. Overall, 17 children achieved perfect scores on every task, while two children (Child 43 and Child 47) achieved low scores on most tasks without achieving the highest possible score on any task. The dendrogram at the top of the heatmap indicates the presence of two main triads of tasks: the pre-assessment tasks (e.g., Pre: T1) and the post-assessment tasks (e.g., Post: T1). This indicates that outcomes for pre-assessment tasks were similar and outcomes for post-assessment were similar. Within each triad, outcomes for Task 2 and Task 3 were more similar to each other than to Task 1, on which fewer children achieved the highest possible score.

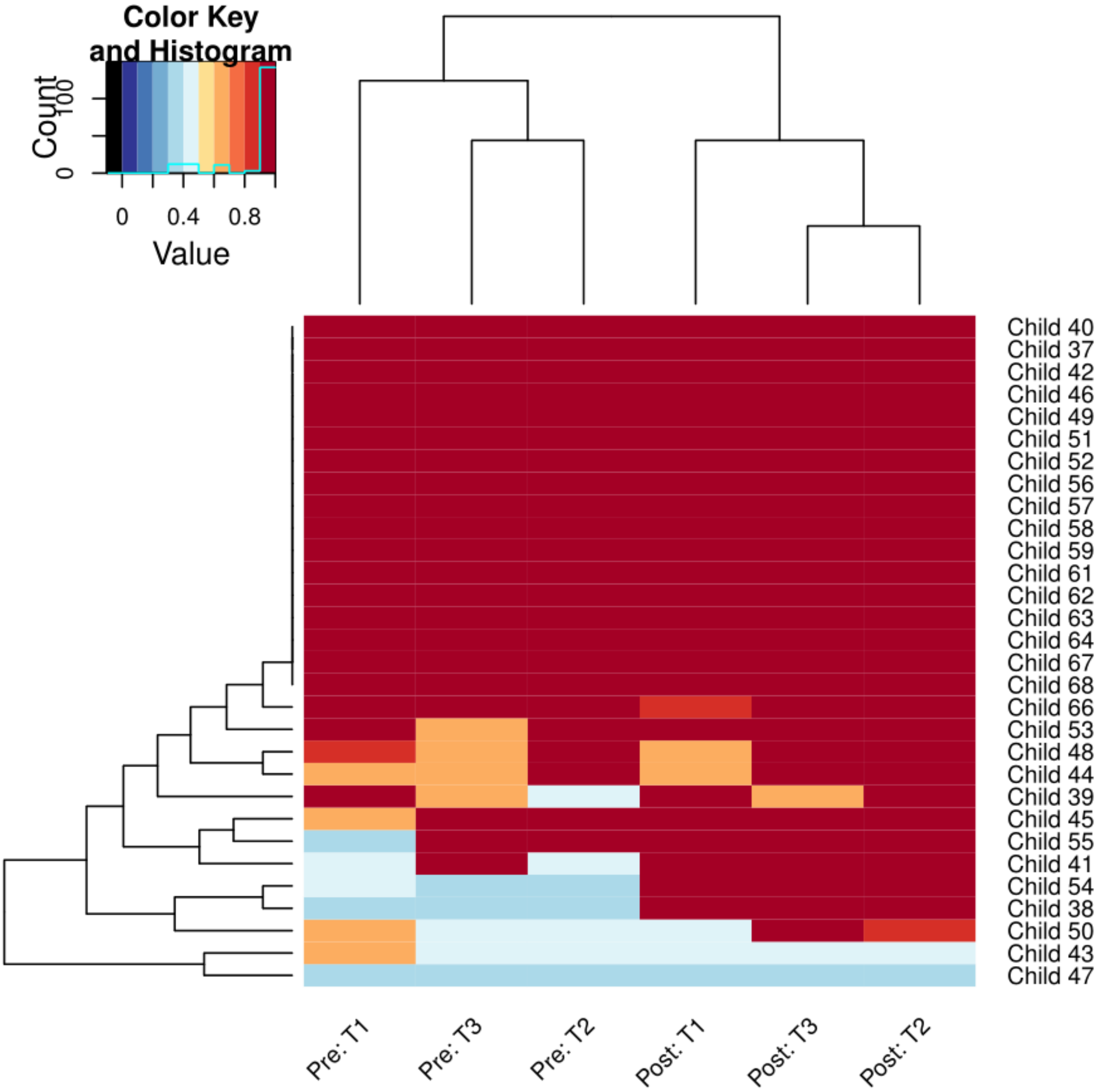

Figure 9. Heatmap and dendrograms of tasks for the pre- and post-assessment in the ENS sequence

Patterns evident in the heatmap included that most children achieved relatively high scores on all tasks and only a few children demonstrated growth from pre- to post-assessment. The heatmaps also showed several unique cases or groups of children, leading researchers to return to these videos to identify patterns in mathematical practices. Several children who did not achieve the highest possible score on a task (e.g., Child 50 Pre: T1) added discs of the same color to those already present and lost count of how many discs they had added. Some children changed their approach during the assessments. For example, on the first two pre-assessment tasks, Child 41 answered aloud first, then modeled the given answer on the ten frame and counted the sum, recognizing it was not equal to the target number. On subsequent tasks, Child 41 modeled before answering. Other children showed growth from pre-assessment to post-assessment, including Child 54 and Child 38. Both children struggled to understand the question “how many more” on the pre-assessment tasks, as Child 54 repeatedly counted all discs while Child 38 tried to add the target quantity of discs (e.g., adding 4 when asked “how many more to 4”). On the post-assessment, both children showed an improved understanding of the prompt; Child 54 used accurate counting strategies and Child 38 added only the required quantity of discs. Other low-scoring children did not show growth. Child 43 often added discs to complete a row of five regardless of the target quantity, or used the same color discs and lost track of how many discs had been added. Child 47 consistently struggled to differentiate between how many discs were on the ten frame and how many discs had been added, even when using different color discs. An overall emergent pattern is that although many mathematical practices and levels of reasoning were possible, few children altered their mathematical practices or levels of reasoning as characterized by the learning progression scores during these interactions.

Hungry Guppy

The Hungry Guppy app: 4-dot and 5-dot Fish activities included app-generated tasks where children combined bubbles showing quantities of dots to make a target number (4 or 5 in this study) to feed the fish (see Figure 10). For example, combining a one-dot bubble and a two-dot bubble created a three-dot bubble. When a child made a combination that was too great for the fish to eat (e.g., 5 for a fish who ate 4's), the bubbles popped instead of combining. The fish grew larger with every correct combination but shrunk if fed too slowly. Each activity involved only one fish with one target quantity. When a child tried to feed an incorrect quantity to the fish, a message appeared and a voice stated that the fish only ate the target quantity (e.g., "this fish only eats 4's"). After the child fed the fish, popped bubbles, or had no way to correctly combine bubbles (e.g., fish eats 4's but only 3's are present), additional bubbles appeared to choose from.

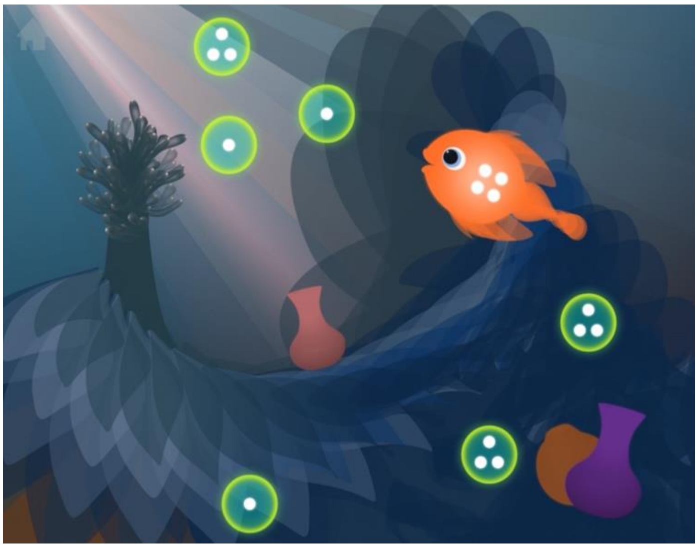

Figure 10. Screenshot of hungry guppy app: 4-dot fish activity

Figure 11 shows a heatmap of the Hungry Guppy results for the 30 children that completed the ENS sequence. The heatmap uses sequential ordering (e.g., Task 1, Task 2, etc.) because children completed different amounts of tasks and the individual tasks varied (e.g., for Task 1, two children could be required to make 4 from different assortments of bubbles). The hierarchical clustering algorithm emphasizes the number of tasks completed, thus it is not useful here to sort tasks by their similarity.

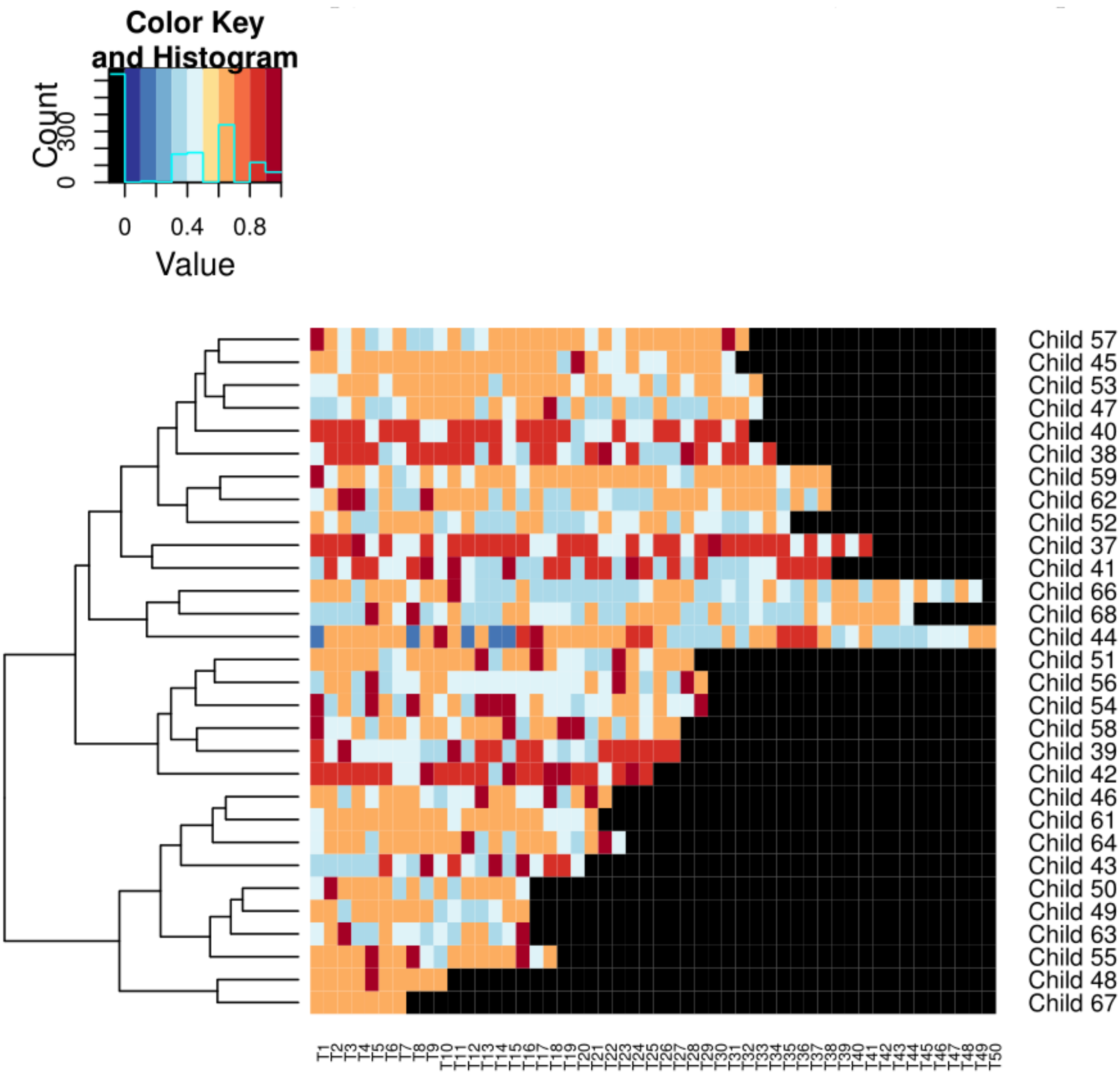

Figure 11. Heatmap and dendrogram of tasks for the hungry guppy learning activity in the ENS sequence

Analyzing the heatmap in relation to the learning progression scores revealed the overall pattern that almost every child consistently applied one preferred mathematical practice. The highest learning progression score (1.0) involved using three addends to make the target quantity without unitarily counting. Only Child 42 and Child 54 achieved this score on at least 20% of their attempts. The second-highest learning progression score (0.83) involved using two addends to make the target quantity without unitarily counting, which at times required waiting until the app generated additional bubbles. The third-highest learning progression score (0.67) involved counting all dots and correctly combining two bubbles. However, while eight children achieved at least one score of 0.83 and 23 children achieved at least one score of 0.67, only Child 44 achieved both a 0.83 (six times) and a 0.67 (23 times). Child 44 was also the only child to score the lowest possible learning progression score (0.17), which involved haphazardly combining bubbles or touching and releasing bubbles without combining. Returning to the videos confirmed that when a child's preferred mathematical practice was possible, the child was unlikely to apply an alternative. Only one child (Child 44) changed between unitarily counting and combining without unitarily counting. Although there were many potential mathematical practices and levels of reasoning for completing tasks, children were generally consistent in their application of a particular mathematical practice when possible.

Fingu

The Fingu app presented tasks consisting of groups of pieces of fruit on the screen, requiring children to indicate the quantity of fruit by simultaneously touching the screen with the corresponding quantity of fingers (see Figure 12). For example, when the app presented two pairs of apples (four apples), acceptable responses included four fingers from one hand, three fingers from one hand and two fingers from the other hand, or two fingers from each hand. The first level involved quantities 0-5, but some children progressed to levels featuring quantities 0-10 in the second level.

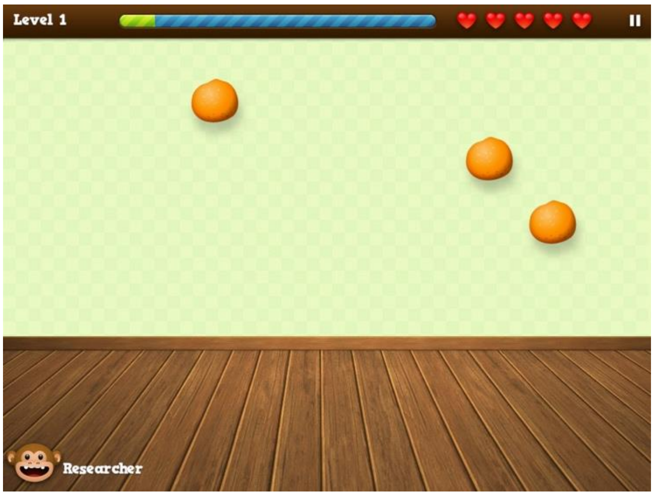

Figure 12. Screenshot of fingu app, Level 1

Figure 13 shows a heatmap of the Fingu results for the 30 children that completed the ENS sequence. The heatmap uses sequential ordering (e.g., Task 1, Task 2, etc.) because children completed different amounts of tasks and the individual tasks varied (e.g., for Task 1, one child might have 4 apples while another child might have 3 oranges).

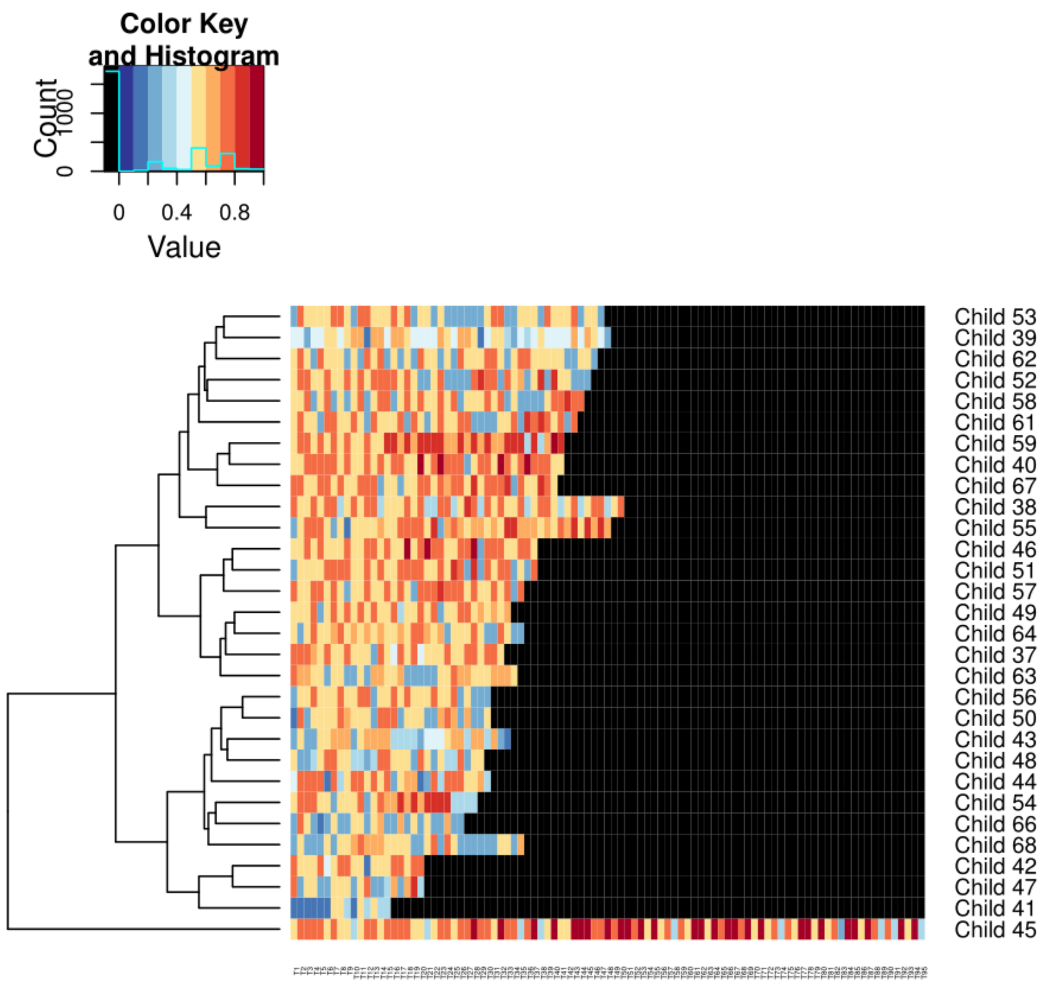

Figure 13. Heatmap and dendrograms of tasks for the fingu learning activity in the ENS sequence

Analysis of the heatmap led to further investigation of patterns and unique cases. Child 45 was an outlier with 95 tasks completed. From the beginning, Child 45 showed advanced subitizing and finger gnosis skills. When two addends with a sum of 1-5 were present on the screen (e.g., 2 groups of apples), Child 45 combined the addends without counting, often indicating the sum based on groups of five (e.g., 2 and 2 indicated as 4 on one hand). Child 45 also applied this strategy when reaching levels involving quantities of 6-10 (e.g., 3 and 3 indicated as 5 and 1). Other children (e.g., Child 59) consistently achieved the highest learning progression score possible for a given task but faced few tasks that allowed for the most advanced strategies. Some children did not consistently apply advanced strategies when possible, such as Child 54, who struggled on tasks involving two addends with a sum of 6-10. Other children heavily relied on counting, such as Child 68 counting when presented with more than 3 pieces of fruit, and Child 39 consistently counting the fruit before indicating the answer, regardless of the quantity present. Some children mixed counting with struggles to effectively indicate an answer. Child 43 repeatedly tried to directly touch the fruit and often used one or two fingers on each hand to input the answer (e.g., 4 indicated as 2 and 2). Child 41 initially attempted to touch each piece of fruit in sequence (e.g., 3 indicated as 1, 2, 3) and progressed to trying to indicate one addend at a time (e.g., 3 and 1 indicated as 3, 1). The overall pattern from the results is that children applied a variety of mathematical practices to complete the tasks, but that most children persisted with their chosen mathematical practices and did not change their level of reasoning.

Discussion

The results of these analyses have implications for learning mathematics and for data analysis. The patterns evident in the interactions align with research on development of mathematical understandings related to early number sense and quantity in base ten, yet showed little change in children's reasoning. Heatmaps serve as visual examples to aid identification of patterns and unique cases, including apparent growth or lack of growth during the interview while identifying whether tasks were appropriate for participants (see also Moyer-Packenham, Tucker, et al., 2015). The coupling of heatmaps with dendrograms, the visualizations of a hierarchical clustering analysis, supports a deeper analysis of not only children's outcomes, but also their performance on the specific tasks within a learning activity. The in-depth examinations shed further light on these areas, including generating questions for further investigation.

Implications for Learning Mathematics

Results of the combined analyses indicated that children used a variety of mathematical practices that reveal their developing mathematical understandings. Many of the mathematical practices and understandings align with mathematical development in established progressions outside of an app context and patterns identified in other studies. For example, while interacting with Fingu in the ENS Sequence, Child 68 and Child 39 relied on counting rather than subitizing, while Child 54 fluently demonstrated subitizing up to five pieces of fruit but struggled with quantities greater than five. Similar to other studies involving children's interactions with the Fingu app, patterns of mathematical practices included attempting to touch the fruit, counting, and immediately recognizing the quantity shown (Baccaglini-Frank & Maracci, 2015; Holgersson et al., 2016). These mathematical practices correspond with elements of established progressions, including *Quantity, Number, and Subitizing* (e.g., Small Collection Namer, Perceptual Subitizer to Five) and *Verbal and Object Counting* (e.g., Counter: Small Numbers, Counter: 10) (Sarama & Clements, 2009). While interacting with Quantity 10-99 in the QBT Sequence, children demonstrated a range of understandings of the value of the blocks within the base-ten system. Approximately half of the children inconsistently used a tens rod to represent ten while correctly answering the prompts, and most of the children who consistently and correctly used a tens rod did not show evidence of fluently combining tens and ones (e.g., 10, 20, 30… 1… 31 vs. 10, 20, 30, 31). Studies involving Quantity 100-999, an advanced version of Quantity 10-99, found that second grade children accurately completed base ten modeling tasks while employing a variety of mathematical practices, with their strategies demonstrating a stronger knowledge of base ten (e.g., fluently modeling using hundreds, tens, and ones) than was shown by kindergarten children this study (Moyer-Packenham, Tucker, et al., 2015; Tucker et al., 2016). These mathematical practices also correspond with elements of established progressions, including *Arithmetic: Composition of Number, Place Value, and Multidigit Addition and Subtraction* (e.g., Composer with Tens and Ones), and *Verbal and Object Counting* (e.g., Skip Counter by 10s to 100, Counter of Quantitative Units/Place Value) (Sarama & Clements, 2009). Future research could examine how patterns in children's mathematical practices and underlying mathematical understandings develop in relation to established progressions across longer-term interactions with touchscreen mathematics virtual manipulative apps.

Although different children exhibited various mathematical practices with differing learning progression scores, the main pattern identified in this study is that most children persisted in their mathematical practices within a set of related tasks. Across the entire QBT Sequence, most children exhibited some understanding of the tens rods as representing the tens place in the target number. However, few children who did not initially demonstrate this strategy (i.e., attained low learning progression scores on the pre-assessment) changed their practices on the post-assessment tasks using Quantity 10-99, instead maintaining their chosen strategy from the first app interaction. While interacting with Hungry Guppy in the ENS Sequence, although children could use two addends without unitarily counting or counting all dots before combining, only one child used both strategies. On many tasks, such as those involving Friends of Ten in the ENS Sequence, most children persisted with a strategy that involved a high level of reasoning and led to success, which may have diminished likelihood of change. This aligns with findings from other studies that suggest children rely on few strategies (or one), and that the idea of varying strategies does not spontaneously emerge (Baccaglini-Frank & Maracci, 2015). However, some children altered their mathematical practices as their understandings changed, such as Child 63's performance across multiple Numerals tasks and the children who began using the tens rod in 1 to 20: 11-20. This corroborates findings from other studies indicating that children's mathematical practices and understandings can change via interaction with mathematics virtual manipulatives (e.g., Desoete et al., 2016; Moyer-Packenham, Shumway, et al., 2015; Tucker, 2016). These results are important because, taken together, they suggest that mathematical practices and understandings that develop during and as a result of interacting with virtual manipulatives can vary. However, this study was exploratory in nature and children interacted with each app for a relatively short time period. Longitudinal or fine-grained investigations may reveal how children develop mathematical practices over time, including factors that contribute to this development.

Implications for Data Analysis

Heatmaps with dendrograms aid identification of areas for further investigation, for both researchers and practitioners. The visualizations supported effective comparison of many participant outcomes, especially when identical tasks allowed direct comparisons (e.g., Task 1 in 1-20: 11-20 always required building 11 from 0 with the same frame present). Initial comparisons for the tasks in the QBT Sequence showed that children demonstrated growth of understanding of the mathematical tasks, as measured by the learning progression scores, on each of the assessment tasks. A quick visual analysis of the heatmap for the first learning activity, Montessori Numbers: 11-20 where most children attained the highest learning progression score, indicates the task was too easy for the majority of the children. In contrast, the Montessori Numbers: Numerals learning activity of the QBT Sequence, the children had very diverse performances with some children completing only two tasks at low learning progression scores to other children completing as many as seven tasks at high learning progression scores. The hierarchical clustering indicated a group of high achieving children and the heatmaps indicated some variations in the performance across the children within that cluster. This prompted the researchers to further investigate the differences in these performance levels. Initial comparisons for the ENS tasks indicated that little growth occurred on the assessment tasks, which might have been too easy for most of the children in this study. However, heatmaps and dendrograms of the learning activities showed mixed results for many of these children. Returning to video data showed that within each app, children were generally consistent in their mathematical practices, though their outcomes varied slightly, often depending on the task presented. For example, the heatmap and dendrogram for QBT: 1 to 20 indicated that Child 65 and Child 36 performed poorly relative to most other children, but returning to the videos revealed that Child 65 encountered difficulty while trying to rush through the tasks, whereas Child 36 did not use a mathematically efficient strategy to complete the tasks. This builds on previous research indicating that heatmaps paired with dendrograms can be effective for visualizing some comparisons and identifying areas for further examination (Moyer-Packenham, Tucker, et al., 2015), with potential to be a valuable component of mixed methods research.

Conclusion

Children's mathematical practices and changes in these practices can be evident as children interact with touchscreen mathematics virtual manipulatives. Investigating patterns and unique cases in children's mathematical practices during these interactions contributes to our knowledge of how children develop mathematical understandings. Children may or may not change mathematical practices and show evidence of increasingly sophisticated reasoning as a result of these interactions. Examining mathematical practices through a combination of qualitative video coding and quantitative data visualized using heatmaps and dendrograms provides a rich view of the data generated by these activities. The visualizations cluster and group the data, facilitating identification of patterns and unique cases to direct focused follow-up analyses and interpretations of the data and what generated the data. For researchers, this may involve returning to the interactions and related tasks, while teachers may consider the student performance and the initial assignments. In each setting, these analyses contribute to analyzing and interpreting children's understandings. Therefore, using heatmaps with dendrograms in tandem with qualitative analyses has potential for enriching our understanding of learning both in research and teaching environments.

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