Semantic Structure of Classroom Discourse Concerning Proof and Proving in High School Mathematics

Isikhan Ugurel[1], Burcak Boz-Yaman[2]
[1]Dokuz Eylul University
[2]Mugla Sıtkı Kocman University

To cite this article:

Semantic Structure of Classroom Discourse Concerning Proof and Proving in High School Mathematics

Isikhan Ugurel, Burcak Boz-Yaman

Article Info	Abstract
	This study tries to identify high school students' knowledge about the concept of proof, based on classroom discussion. The processes of discourses, both natural and prompted, are studied as they occur between students and teachers. The study employs discourse analysis as the qualitative research framework. Participants are 13 Science High School students from Izmir (Turkey) in 11th grade and two mathematics and geometry teachers. Data gathered consist of 53 filmed mathematics and geometry classes, recorded over three months, plus researchers' field notes. The focus of the study is on verbal discourse in the classroom between teachers and students. 18 recorded discourses were analyzed after transcription. The theoretical framework of the study is dependent on the social semiotics, with Halliday's Systemic Functional Linguistics model (SFL) used in the analysis. In the SFL model there are three main components; field of discourse, tenor of discourse, and mode of discourse. This study presents findings and analysis results based on the field of discourse. Emergent findings showed important effects of teacher-student in-class discourses (in terms of the structure, diversity, and pattern characteristics) on the students' learning about proof and knowledge constructions.

Introduction

The multi-dimensional relationship of mathematics with linguistics, and its linguistic identity, is an important research topic, not only for mathematics, but also among diverse disciplines such as philosophy, psychology, sociology, semiotics, educational sciences, and mathematics education. In-depth investigation of the relationship between language and mathematics provides the opportunity to gain profound and substantial insight into the structure of mathematics, facilitating more qualified teaching of mathematics at every level. It is accepted that language itself takes precedence over concepts among existing, thinking, understanding, learning, and social entities of the human being. From this point of view, investigating the functions of language during mathematics teaching and learning, and the effects of formal language on learning mathematics, provide for rich theoretical approach and methodological framework for research about doing mathematics, understanding mathematics, how learning occurs and improves, and the inquiry into the mechanism of mathematical thinking. In this so-called communication era, it is necessary to possess skills of constructed, healthy and progressive societal communication, both individually and also among others in certain social structures.

Nowadays, with broader, intensified communications, reconsidering the approaches to communication is also relevant in education, and this leads topics of research on the existing forms of communication in the learning environment, with a focus on their effects on learning and understanding. Cazden and Beck (2003) suggested the quality of classroom discourse as the primary topic among school reform debates at the end of the 20th century. Many research studies verified this argument in the literature. The National Council of Teachers of Mathematics (NCTM, 2000) emphasized in their principles and standards, that communication is an important aim in mathematics teaching and learning, from kindergarten to high school; all students should have the opportunity to arrange and reinforce their mathematical thinking with communication, and communicate through mathematical thinking with peers, teachers and others. The National Council of Supervisors of Mathematics (NCSM) considered it necessary to include communication with mathematical ideas among the 12 components of successful mathematics education (Ellerton & Clarkson, 1996). In South Africa, the 2005 Mathematics Education Curriculum emphasized language as playing a vital role in learning mathematics, stating that "learners should use the mathematical language to communicate about the mathematical ideas, concepts, generalizations and thought processes" (Department of Education, 1997, as cited in Setati, 2005, p. 77). The high school mathematics curriculum of the Swedish National Educational Agency underlines students' group

works and their mathematical communication skills (Ryve, 2004). Kharing, Hamaguchi, and Ohtani (2007) stated in the Japanese Mathematics Education Curriculum that although there are no obvious standards about communication, Japanese mathematical communication is strongly developed and emphasized in Mathematics Education. In the Taiwanese Mathematics Education Curriculum, Lin, Shann, and Lin (2007) stated in the 2003 education curriculum where in four main topics, there are nine objectives related to communication. Similarly in the Curriculum of Singapore (Har, 2007), Peru (Miyagui, 2007), Philippines (Ulep, 2007), and China (Wang, 2007), mathematical communication is emphasized in mathematics education. In the Turkish mathematics curriculum of 2005-2006, developed with a new understanding, communication is a central skill and clearly underlines the necessity to communicate mathematically when learning mathematics. In the conceptual structure of the curriculum, communication is one of the six standards. Likewise, in the current 2013 Mathematics Curriculum, one of the mathematical process skills is using mathematical language and terminology effectively and correctly (referring to mathematical communication).

> Talking, writing and listening about mathematics is improving the communication skill and at the same time it helps students to understand mathematical concepts better. The teacher should construct a classroom environment where students could explain and discuss their thoughts and express them by writing, and also she/he should conduct proper questioning to make students communicate better (MoNe, 2013, p. VII).

It is observed that an internalization of curriculum approaches in the institutional education documents is highly affected from the improvements on research areas and cumulative literature. Sierpinska (2005) stated that during the last 20 years, research about language in mathematics education involves at least three common theoretical approaches with language as a code, representation, and discourse. Hiebert, Carpenter, Fennema, Fuson Wearne, Murray, Oliver and Human (1998) claimed that students accomplish mathematical understanding only when they form connections between knowledge and the relations/links of their knowledge; so that communication is the key component to providing the associational understanding (cited in Steele, 2001). Methods of investigation about communication and its theoretical source is offered in the disciplines of language, psychology, and sociology. Social sciences provide not only discipline-specific information, but also the opportunity to apply interdisciplinary original research to investigate language and the related processes of communication. One of these areas is discourse analysis (DA), which can be explained as a theoretical framework and qualitative research method with specific steps, and an interdisciplinary science that encompasses other disciplines and combined analytical approaches in order to analyze language from different perspectives. The common usage and evolution of DA among the social sciences make it feasible for mathematics education, where communication is a basic competence of mathematics teaching and learning.

According to the literature, some research investigated communication as linguistics, with most related to primary education (Huang, Normandia, & Greer, 2005). Most studies investigate mathematical language in mathematics education and mathematical application, focusing on words, symbols, and isolated special grammatical forms (Morgan, 1996, as cited in O'Halloran, 2000). Among the research, there are few studies that investigate students' learning, understanding and abilities on proof and proving through analyzing classroom discourses as communication. Proof and proving is central to both enhancing mathematical thinking (including advanced mathematical thinking) and doing mathematics. This applies in understanding the structure and nature of mathematical knowledge, comprehending the historical development process, understanding mathematical objects, ways of developing knowledge, and how individuals and society share mathematical knowledge (Uğurel, 2012).

Within the scope of such research, proof is discussed with more extensive content, taught through various methods and approached more intensively in school mathematics as part of reforms within mathematics education. After reforms to many mathematics curricula, within NCTM (2000) standards, and in Common Core State Standards for Mathematics (CCSSM, 2010), reasoning and proof is considered a basic area that should be taught at all levels of education. Thus, in the current study, DA on the basis of linguistics tries to fill a gap in the literature on the discourse perspectives of proof and proving. In high school classrooms, the study looks to discover how secondary school students structure their knowledge about proof and proving through discourse characteristics, based on in-class communications. This current study is perhaps pioneering in mathematics education, employing DA as a qualitative research method. In the study it follows the functionalist linguistic perspective, while analyzing discourses during communication. In the theoretical framework, the functionalist approach and social semiotics, which constitute the study's framework, are discussed. This paper reports a part of the large scale research project which is focusing on the discourse of whole-class interaction in mathematics classroom. The research question for the main project is: "How secondary school students construct their mathematical knowledge about the concept of proof by helping in-class communication patterns?" To give an

answer to this question DA is conducted depending on Systemic Functional Linguistics theory to investigate discourses on proof in the classroom. This paper specifically focused on the 'field of discourse' about proof during in-class communication, with results and detailed findings discussed.

Theoretical Framework

Actually all kinds of DA originate from the non-educational field and most are related with linguistics (Rogers, Malancharuvil-Berkes, Mosley, Hui, & Joseph, 2005). Although other disciplines created diversity through DA from past to present, linguistics is clearly dominant. Linguists have improved two different approaches to language. The first one emphasizes grammar, and the other one accentuates natural ways of using language with specific aims in specific communication environments. Linguists in the first group are called formalists, whereas the second group is known as functionalists. The formalist approach is concerned with the grammatical properties or structural perspectives of language, whereas the functionalist studies how the language is used (Erton, 2000). Formalists (e.g. Chomsky) tend to deal with language as a mental fact, while functionalists (e.g. Halliday) view language as a social fact (Leech, 1983). According to the functionalist approach, language is the source of generating meaning (Huang & Normandia, 2007).

Starting in 1930, the approach of linguistics was mainly structuralist, but from 1980, functionalist studies increased; discussed under two linguistic domains, semantic and pragmatic. It's possible to create a formal representation (Figure 1) by associating approaches dealing with the relationship among structure, meaning, and discourse in the English language from a linguistic point of view, with formalist and functionalist paradigms. Using different theoretical perspectives and analytical approaches, analyzes can be conducted by helping discourses which investigate meaning, configure the components and examine the general mechanism; e.g. speech act theory, interactional sociolinguistics, ethnography of communication, pragmatics, conversation analysis, variation analysis (Schiffrin, 1994), and content analysis (Bilgin, 2000). For functionalist linguistics, there are also different theoretical frameworks that examine meaning in context. Among these frameworks is social semiotics, which forms the basis of the current study.
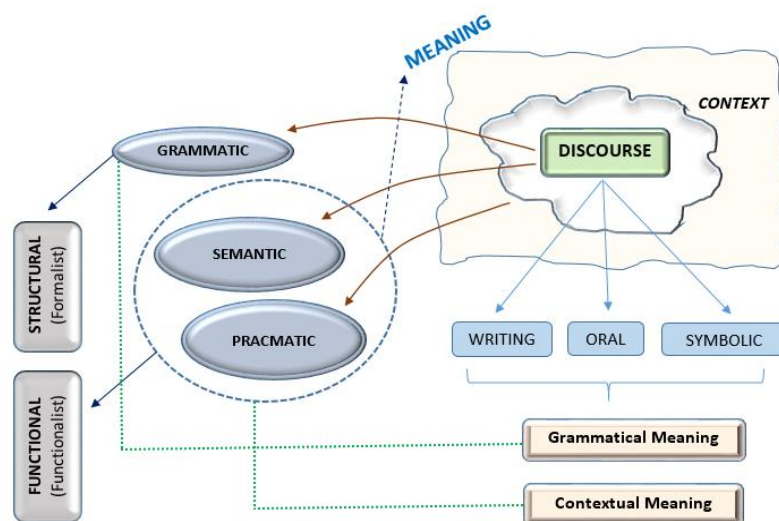
Figure 1. Relationship discourse-meaning-sentence regarding linguistics[*]

Social Semiotics

The core of social semiotics is that meaning is constructed/created, i.e. meanings do not exist as objects or concrete reality, but are constructed with indicators/signifiers (symbols) (Chapman, 2003). The indicator/ signifier as a concept is a symbolic form used for reflecting people to themselves and others, anytime or anywhere (Günay, 2004, p. 56). Paintings, photographs, traffic signs, gestures and facial expressions, colors, music, and written and verbal words are some examples of signifiers. Underlying social semiotics is an examination of the meaning, depending not purely on linguistic relationships between signifier and signified (the represented psychical object) (Chapman, 2003). One thing that social semiotics provides is

---

[*] While this figure is producing, two linguistics from English and French Literature Departments are consulted.

conceptualization of the context itself (Morgan, 2006). According to social semiotics, context refers not only to the instant/situational context when communication occurs, but also to more expanded cultural context which contains itself. In its most general sense, context can be explained by "the events that occur around when people talk and write" (Halliday, 1991, p. 5, as cited in Celce-Murcia & Olshtain, 2000, p. 11), or the universe produced by the people who are talking (Zeybek, 2003). "Discourse linguistics believes that linguistic choices are not random, but decided by contextual factors systematically (He, 2002, p. 4)". Cem (2005) stated that linguistic structure has three dimensions; structure, meaning and usage, emphasizing that linguistic structures/artifacts are produced by form and that rules are necessary in appropriate contexts to have meaning in communication. Moreover she explained that;

> The dimension of meaning expresses the purport of linguistic structure, the usage on the other hand, shows in which context and by whom and in which places that structure is used and explained the assumptions about these contexts (Cem, 2005, p. 11).

Systemic Functional Linguistics/Grammar

DA studies are the core of Systemic Functional Linguistics, and Halliday is one of the most important English linguists, whose framework emphasizes the social functions of linguistics (McCarthy, 1991). According to Halliday, the focus of related studies is the usage of language in daily life to accomplish a communicational aim (Eggins, 2004). Halliday's linguistic model, called Systemic Functional Linguistics (SFL), has considerable influence (Fawcett, 2000). "This model that get constitutive from social semiotics, it is accepted an important descriptive and interpretative framework which language is regarded as a strategic and meaning making source" (Eggins, 2004, p.1-2). SFL first defines language not as a system or rule, but the source of the meaning. Second, it does not consider sentences, but text as the basis of the grammar, i.e. it approaches grammar to realize the discourse. Third, SFL investigates common relationships between text and context (Halliday & Martin, 1993). SFL suggests four basic theoretical claims about systemic linguistics;

> 1-Using language is functional, 2-The function of the language is making a meaning (producing meaning), 3-These meanings are effected from the cultural and social context with which the meaning is exchanged, 4-The process of the using language is semiotics process (the process of meaning making by doing appropriate selection to the context) (Eggins, 2004, p. 3).

With regard to social semiotics, every contextual situation may be considered as an example of constructed semiotic structures, but not an isolated social-semiological variable (Morgan, 2006). These variables are called field of discourse, tenor of discourse, and mode of discourse. Those concepts are improved by Halliday to investigate and interpret the discourse where they describe three perspectives of social context (Atweh, Bleicher, & Cooper, 1998).

Field of Discourse (FD): It deals with what's going on in discourse and the nature of the action (Atweh et al., 1998). In another words, FD is about the language used, the experiences of the participant, and what happens through language (Benson & Greaves, 1981). FD may be related with different kinds of social actions such as doing the dishes or discussions in parliament (Renkema, 2004). If text is analyzed regarding FD, lexical items are first investigated (Mechura, 2005).

Lexical items: It can easily be understood what a text is, related to when the lexis are investigated and the kinds of specific lexis within the discourse. When a discourse is investigated through DA, two basic questions about lexical items are considered (Mechura, 2005).

> a. In the discourse, to which field the lexis belong?
> b. How well do the general audience and experts in that specific field know about the lexis in the discourse?

These questions involve investigations; the first looking at the semantic field, and the second, specialization. In the semantic field specific words (e.g. names) can be searched for, or checked to a dictionary related with the discourse. For specialization, researcher intuition comes into play, as does the discipline-specific dictionary/list of words or the corpus research about the discourse (Mechura, 2005). In the current study, the findings are presented only about the field of discourse.

Tenor of Discourse (TD): This relates to who is participating in the discourse, the nature of the participants, and their status and roles (Atweh et al., 1998). TD "produces clues about the relationships of the roles among the participants" (Renkema, 2004, p. 46).

*Mode of Discourse (MD): "*The mode could be described as interpreting how the participants accomplish these activities through the nuances of the language used" (Atweh et al., 1998, p. 66). MD provides answers to questions about how discourse is constructed and how it is transmitted (Mechura, 2005). MD answers the following questions; How is the organization of the discourse? What is the function of the discourse in the context?, and What is realized by the discourse? (Renkema, 2004).

It can be seen that social semiotics is very important in linguistics and also enters the study field of mathematics education. Morgan (2006) claimed that Halliday's social-semiological theory not only presents a strong strategy to investigate mathematical and teaching-learning practices, but also helps to produce knowledge towards using language that contains mathematical practices that assist teaching-learning. In mathematical learning and while improving an approach to linguistics, social semiotics revealed a very complex, mutual relationship among the ordered four constructs of cognitive, linguistic, social (interaction), and context which seems different in appearance (Chapman, 1993). Many research studies agreed mathematics is a semiotic area (e.g. Radford, Schubring, & Seeger, 2008; O'Halloran, 2004; Straber, 2004; Marks & Mousley, 1990). Therefore, in mathematics education it is possible to use semiotics and social semiotics approaches about the role of symbolic, visual and linguistic signifiers in mathematics education; the effects of them on learning; and on research about mediator roles/duties that realize mathematical understanding. However, according to the literature, research is relatively scarce, especially about social semiotics. Among these studies are research that used Halliday and his colleagues' model (FD, TD, and MD). Some research used SFL (e.g. Atweh et al., 1998; Thornton & Reynolds, 2006; Herbel-Eisenmann & Otten, 2011; Herbel-Eisenmann, Johnson, Otten, Crillo, & Steele, 2014).

Atweh et al. (1998) examined mathematics teaching, and how teachers' perceptions about students' abilities and expectations are reflected in classroom discourses, based on socioeconomic status and student gender. For this purpose, two high schools of different socioeconomic status; one of which was girls-only and the other boys-only, were selected. From both high schools, two 9th grade classes in which the same topics (functions, linear equations and drawing graphs) were discussed and used the same course book were observed for ten lessons. Then, both teachers' classroom discourses were analyzed through three components (FD, TD, MD) of Halliday et al.'s model.

In another study, Thornton and Reynolds (2006) produced critical discourse analysis (CDA) which contained SFL. In this study, discourses from an 8th grade mathematics classroom were analyzed through Fairclough's (1992) three dimensional CDA framework. The classroom teacher and the students had a friendly classroom environment which was both comfortable and conducive for study. The topic of the observed class was equations with y=ax+b form and examining the changes with respect to a, finding the slope and drawing the graphs.

Another research study which investigated classroom discourse was that of Freitas and Zolkower (2011), which was conducted over two years through lesson studies with 12 middle school teachers. They investigated the key social semiotic concepts, focusing on the complex conjunction of the mathematics register and everyday language. In one classroom, the discourse about a non-routine problem called the 'fishpond problem' was examined through linguistic and diagrammatic challenges by using three simultaneous meta-functions of SFL (interpersonal, ideational, and textual).

In another study, Herbel-Eisenmann and Otten (2011) examined mathematics discourses by using SFL, by using it especially on the field of discourse to understand how mathematics is constructed. Moreover, Herbel-Eisenmann et al. (2014) investigated discourses on mathematics register and artifacts (e.g., posters, reports, diaries) generated by secondary school mathematics teachers by helping SFL. Nine mathematics teachers participated in the study group for one year and mathematical discussions were conducted based on mathematical discourses through practical pedagogical development materials. The researchers showed that the teachers' communication and the establishment of meaning of the mathematical register shifted over time.

Different from other classroom discourse studies, the current study sheds light on the process of understanding proof and proving concept through discourse analysis dependent on SFL in a high school mathematics classroom. Therefore, the current study should be considered as a contribution to the related literature and especially in Turkey, in order to gain the attention of mathematics educators to this concept.

Method

Participants of the Study

In the current study, the classroom is the field of study as proof is sufficiently large a topic within this sphere. With this in mind, information was gathered by conducting pre-interviews with mathematics teachers from four different types of high schools (vocational, normal, Anatolian, and Science High Schools) in Izmir regarding how much time they spend on theorems and proofs, and what are the general attitudes of students' on those topics. According to these pre-interviews, Science High Schools were found to give more time on proof and proving, and because of this, it was decided that a Science High School be chosen as the field of study. Research employing DA is generally conducted on small groups, and because of that, the selected Science High School classroom should not have many students and should have healthy levels of communication.

Following face-to-face interviews conducted with administrators and mathematics teachers of Science High Schools, a private Science High School was chosen from the central district as appropriate for the study. From these interviews, it was learned that proof and proving are taught in mathematics and geometry classes as part of the curriculum, and tested within central exams. Some of the 11th grade students are partially accustomed to proofs because they attend supportive seminars about TUBITAK project studies. From a briefing about the aim and scope of the current study, Ayşe, a mathematics teacher, stated that their 11th grade Science class has 13 students at almost the same achievement level, and would be eager to join the study and are open to frank communication. It was therefore decided that this class would be selected for the current study. The class is taught mathematics by Ayşe and geometry by Ahmet (pseudonymous assigned). Since the aim of the study is to observe in-class discourse for both subject areas, as well as communication between students and their teachers, 15 participants were selected for the study; 13 students (one female, 12 male) and two teachers (one female [mathematics], one male [geometry]). Both teachers graduated from the Science Faculty and had worked in private courses (known in Turkish as 'Dershane') for a short time. Ayşe also graduated from a Teacher Training High School and had worked in different public high schools for almost 30 years, until her last assignment at a state Anatolian High School. Since retirement from state education teaching, Ayşe has working at the private school (where the current study was conducted) for nine years, where she is head of mathematics. Ahmet worked in a company for a short time before graduating from the mathematics department. After that he worked on private courses before joining the same private school five years ago. Ahmet obtained a non-thesis master's degree and is the teacher responsible for the TUBITAK Mathematics Project study group/team.

Research Site and Classroom Culture

According to both observation of the researcher and explanations of mathematics teachers, the Class 11-Science is a typical Science High School class, with academic achievement, study performance, and career goals very high for the students. The majority of the students (n=11) preferred individual studies, although they were talkative, extrovert, and easily expressed their ideas, questions and defended their ideas. The students who were successful at primary school level are still considered very successful students. The classroom environment is very competitive; however, this competition is deemed appropriate and healthy, is considered the norm and not detrimental to classroom communication and social sharing. To improve and maintain this kind of understanding, both students and teachers share a common approach. Competition was based on solving easy and mid-level questions very fast, and for more demanding questions, to find the answer or to understand the original solution, as well as seeking the top positions concerning common school exams or at private courses. During the learning of a topic there is no competition, and students share and help each other. The ultimate aim of both teachers and students is the achievement of high central exam scores. This aim leads to teachers and students behaving differently. Both of the teachers stated that students have fast understanding ability and also they have neat, systematic working habits. Having these abilities and a secondary school mathematics curriculum not considered good enough (science high schools have no special curriculum), was seen as an excuse not to fully implement the newly reformed curriculum. The new curriculum is only used for general syllabus, and activity-based learning is not applied in class pairs or small group studies. The teachers produced their own format for order and presentation of the topics to be taught. Especially Ayşe, the mathematics teacher, aimed at conducting very fast basic exemplification and lecturing, giving more time for examples and problem-solving with variety in numbers and types of questions. Since students adapted well to this kind of lesson, topics were finished ahead of plan, and in the remaining time Ayşe teaches from the next year's schedule (for quick recaps and short introductions), or repeats previous topics. Both teachers used their one-hour mathematics and geometry courses for studies based on central exams. Students can also learn mathematics topics from their private course teachers, or from extra studies to solve questions not from the curriculum. In their classes, both

teachers avoided teaching lessons from set textbooks. Ayşe only uses the secondary school level textbooks prepared by the Ministry of National Education (MoNe) to a minimal degree.

The teachers improve their own lecture notes from their experiences and from other books and materials. The same happens for the solving of questions and problems. Homework assigned to students is selected from four or five different books (mostly exam-dependent), according to their appropriateness to the topics being taught.
In Ayşe's lectures, she conducts the teaching herself. Much of the lesson time is assigned to knowledge about mathematical basics, where lecturing is kept to a minimum and more time given over to students' questions or extended in order to meet students' expectations. After that exercises and problem-solving are conducted, generally starting off with average difficulty questions, followed by more difficult ones. At the end of each lesson, homework is assigned to students, and at the beginning of the next class homework is individually checked. Homework mostly involves solving questions. Where students experienced difficulties, they are supported with additional information about questions, rehearsing the topic, giving more examples about problems, solving similar questions, or going over solutions together more than once. The questions that students couldn't solve are discussed in the classroom and solved through contributions of other students.

Ahmet also lectures by himself, with lecturing processes almost matching Ayşe. On the other hand, Ahmet gives more time to students for making explanations, doing solutions, understanding and explaining the solution. He listens more to the students and tries to make them speak out and discuss. Besides, Ahmet sometimes conducts timed tests that are produced by himself in his classes. Neither teacher really uses technology or different teaching-learning tools in their lessons. Sometimes Ahmet applies tests in the technology classroom using the smartboard. Different kinds of homework like investigating, preparing presentations, and performance homework are assigned to the students, except for problem-solving. Both teachers measure academic achievement of students, yet they do not apply any effective domain observation or measurements. The types of measurement instruments used are based on individual grading of written exams, oral exams, and test applications.

Data Gathering Process

The selected 11-Science Class has five hours mathematics and three hours geometry each week. Both teachers dedicate one hour every week for central exam studies. Those two hours do not cover proof or theorem, and were therefore excluded from the study. The secondary school students' in-class verbal discourses (student-student, teacher-student) within the geometry and mathematics classes are the data of the current study, with discourses captured by video recording. The discourse data that explains the process is divided into two categories; Natural Discourse (ND), and Prompted Discourse (PD); considered to be the effect and context of the discourses.

ND represents in-class discourses that happen in mathematics or geometry classes produced during communication between students-students, teachers-students about verbal, mathematical, symbolic or combination of them towards proof and proving; without researcher influence. PD consists of discourses with verbal questions posed in the classroom, prepared by the first researcher in advance, and presented to the mathematics and geometry teachers. The aim was to seek students' explanations, through discussions and the sharing of ideas about proof and proving. There are two aims of PD; the first being to construct a base for analyzing and interpreting the ND; second, to improve chances of producing an environment involve speaking and discussing proof and proving within the classroom, where students can communicate in an environment that helps to release the ideas through discourse. For PD there were 25 open-ended questions prepared to illicit verbal answers (see Appendix). However, instead of asking all 25 questions, the teachers were asked to use only some of them, as appropriate to the situation. While preparing the questions another specialist on mathematics education was consulted and some necessary arrangements were conducted based on his/her suggestions. The researchers did not provide any answers of these questions, nor did they give any information or explanations. These questions are produced based on the extended literature, depending on proof and also classroom observations conducted before the applications. All decisions and preferences are left to the two teachers regarding the verbal questions prepared for PD - used by which teacher, in which conditions, in which context, (individual or group), answered only by students, including the teachers' answers, to start each class, or just at the beginning or end of a specific lesson each week, or according to the curriculum and depending on teachers' opinion etc.

Apart from some encouragement to produce PD involved discourses with the help of these questions, it was attempted to let these processes develop naturally, i.e. for PD to occur depending on classroom norms shared by

social constructs produced in a micro classroom culture by teachers and students; and also aimed to provide participants with a role in the communication. Wood and Kroger (2000) pointed out that data is named as invented discourse, as produced by the researcher, and can be used for evaluating previous research, producing theoretical argument and perspective, helping analysis or supporting an experimental claim. Thereby PD's are produced and evaluated with reference to the idea of invented discourse. In mathematics education literature, no other study which used these kinds of discourse groups was encountered. Therefore the current study is the first to investigate discourses involving both ND and PD through their interactions. The data was gathered over three months during the spring-summer semester from March to May. The first author was present in every class as an observer, and took field notes while recording the lessons.
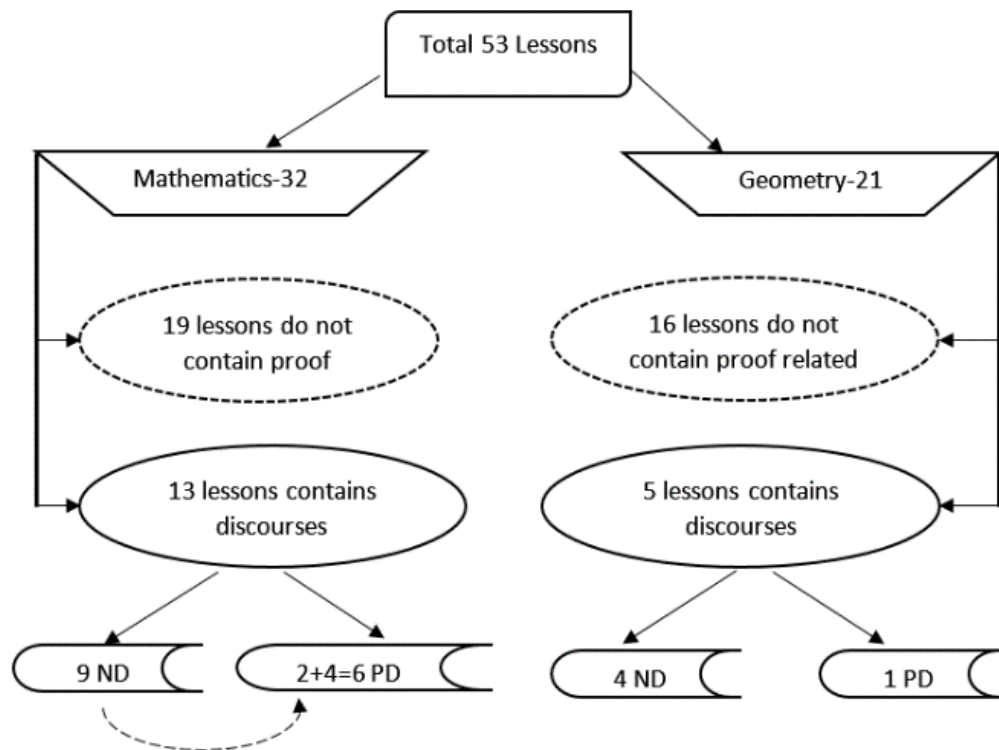
Figure 2. Numerical separation of lessons with respect to proof and types of discourse

Over three months the first researcher collected and video-recorded 32 mathematics and 21 geometry lessons, of which 18 had discourses involving proof and proving (13 mathematics, five geometry). Among the lessons with ND, there are 23 proofs. Among 13 lessons only seven of them are purely ND, four of them are purely PD, and two of them (ND-9/PD-1 and ND-10/PD-2) involved both ND and PD. These two lessons are 33rd and 37th lessons among 53 lessons. In these lessons, Ayşe asked questions prepared for PD and then continued with the lecture for the remaining time. Because of this, these two lessons appears as both ND and PD. In the 21 geometry lessons there were five lessons with discourses involving proof and proving, four of them were ND and one PD.

Transcription

The data gathering was followed by the transcription. Before the transcription began, a search was conducted about the context and format of the recording in light of the literature, the research questions and the coding system used. Researcher watched the recorded videos, taking observations notes from beginning to end. In the current study, gestures and facial expressions, toning and pause times between discourse and theoretical investigation of grammar are excluded from the transcription process. The first author then transcribed all 18 of the recorded lessons with discourses involving proof and proving.

Figure 3. An example of transcribed text

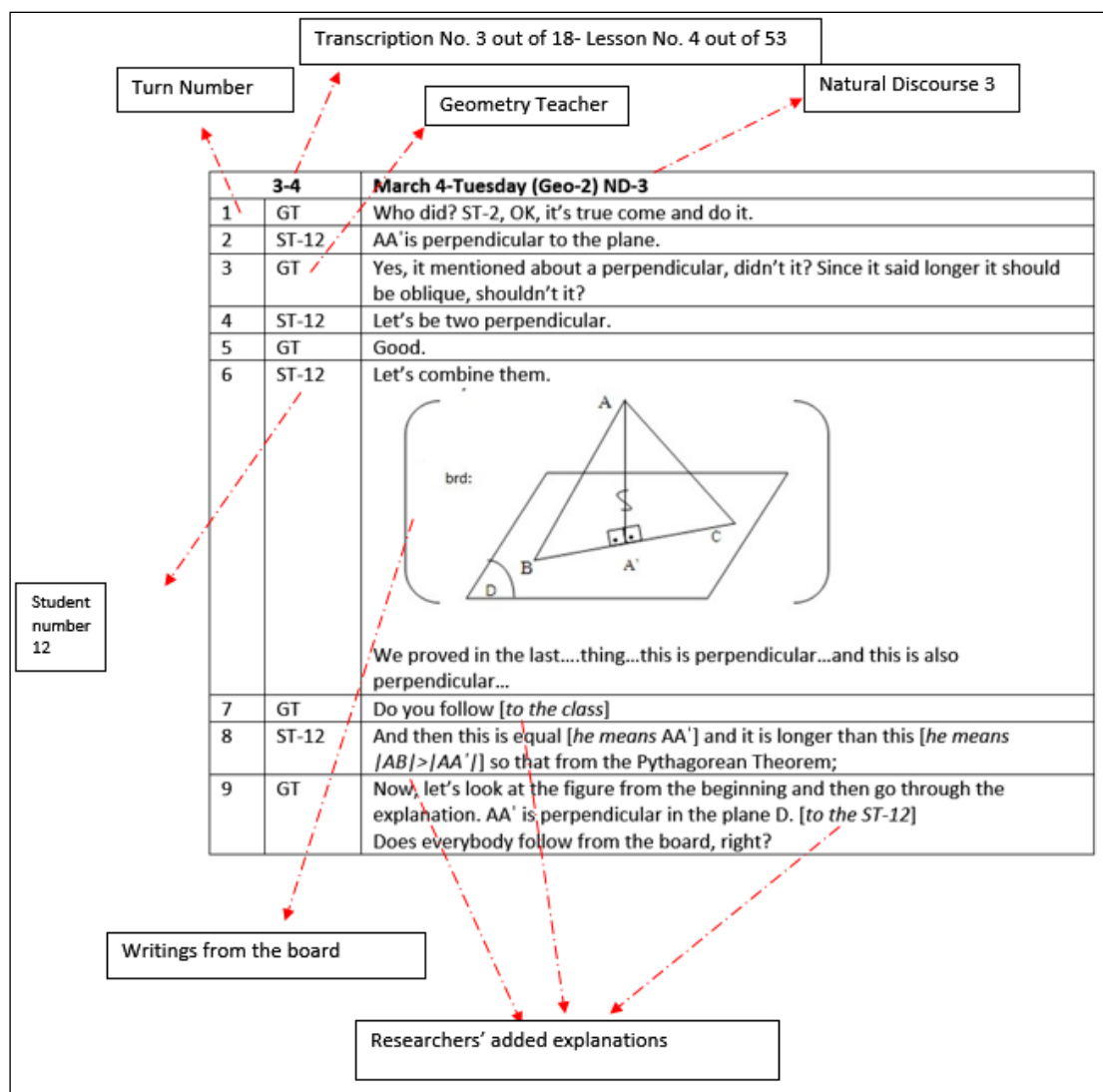During the investigation of the information about proof in the classroom, the discourses constructed by formal or informal, written (on the board) or verbal, symbolic, monolog or dialog, were considered as data for the study. Therefore, during transcription it was decided to use ordered prosaism. First researcher produced two writing draft formats considered appropriate to use. These drafts were then presented to two experts in order to garner opinion about the correctness of transcribing the recorded videos. With some changes according to their feedback, the second draft format was agreed. Each verbal statement was numbered and coded according to the participants (MT, Mathematics Teacher; GT, Geometry Teacher; ST-n, the nth student; SS, Some Students).

Statements are numbered according to priority-recency, the turn number of every statement, and the characteristics of the discourse. Moreover, in the lessons, information written on the board by students or teachers are indicated as "brd: (board)" in parenthesis. Where necessary, additional explanation from the field notes are shown, italicized in parenthesis. Additional information in the transcriptions show the lesson number among the 53 total lessons observed, the lesson number among the 18 with proof involved (transcribed) lessons, the date of recording, course type (geometry or mathematics), the lesson hour, and discourse types (ND/PD). To aid better understand, a selected transcript text is shown in Figure 3.

Results and Discussion

The findings are presented under two topics. First, there are communicational linguistic pattern characteristics and examples towards proof and proving from the field notes taken while recording the lessons. Second, 'lexical items' suggested for analyzes of FD in SFL, are discussed with tables presenting discourse excerpts.

Characteristic Patterns in Teachers' Discourses

In both classes, FD towards proof in general involved discussions on mathematical topics, and interpretations on conducted proofs in particular. Therefore, according to FD, discourses included many themes of mathematics and geometry, proofs as taught in mathematics topics, mathematical representations, and their meanings and practical applications.

In mathematics, Ayşe exhibits the same approach towards proof and proving with other lessons. According to field notes, based on FD, the patterns of characteristics in discourses involving Ayşe are as follows:

- Early on, Ayşe provides the statement (verbally or written on the board) that should be proven;
- No waiting time was allocated for students to attempt the proof themselves;
- Ayşe either starting to make the proof immediately (on the board), or directly told students how to conduct the proof and each next step;
- There were multiple interferences, even dictating to the student selected to attempt the proof on the board in front of the class.

Examples:

ND-1: (11), (18),(21), (23),(24)		ND-9: (229),(231),(262),(270),(326),(352),(354)
ND-4: (15),(18),(25),(32),(36)		ND-10: (109),(140),(165),(184)
ND-5: (11),(13)		ND-11: (8),(23),(101),(102),(114),(123),(127),(129)
ND-8: (13),(19)		ND-13: (28),(30),(39),(53),(54),(69)

15 ND-4	**MT**	Yes, very beautiful! Now, matrices have commutative properties for addition, I mean A+B=B+A. How we can show this, how can we prove it? Let us think generally. Instead of these matrices [*meaning A]* in the kind of $[d_{ij}]_{mxn}$ plus $[b_{ij}]_{mxn}$ are equal.
		First look at this side [*showing the left side of the equation*]. How do we attempt the proof?
18 ND-4	**MT**	Yes, we get the other side by using one side of the equation. Equal, with the kinds of $[a_{ij} + b_{ij}]_{mxn}$. Now, you know the addition, the inside addition was normal, has a commutative rule. I wrote $[b_{ij}+a_{ij}]_{mxn}$. From this point, can I write this statement like this $=[b_{ij}]_{mxn} +[a_{ij}]_{mxn}$? If I can write that, this is B matrices and that is A matrices.

The dominating emphasis from Ayşe about the functions of proofs and the aim of proving and proof is to provide the opportunity for practical applications and computational rituals. Because of this, in her discourses there are statements towards providing a formula, rule or shortcuts through proving.

Examples:

ND-1: (48),(59),(64)		ND-8: (13),(19)
ND-5: (1),(49),(50)		ND-9: (229),(309),(310)
ND-6: (25)		ND-10: (151)

59 ND-1	**MT**	From this we get a usable formula. That's okay kids! [*waits 10-15 seconds*]
151 ND-10	**MT**	We separate k+1 and k! and then write one by one, it is simplified at last, and we can show that from 1 to n k is k.k! which is same with (n+1)!-1. In that case, can we use these kinds of questions? What if I asked, what is this $\sum_{k=1}^{100} k.k!$ and look at this (n+1)!-1, you can write 101!-1 and use the proven element.

Ayşe does not allocate extensive time for making the proof. She conducts proofs very fast and aims at showing the exemplification of the constructed formula or rule, and continues on with applications on some problems.

Examples:

ND-1: (5),(45)   ND-4: (70),(74),(83)   ND-10: (151)

70 DS-4	MT	The rule is that, now you can see when we do exercises, why it is so, and that it cannot be multiplied in another way. We multiply the two matrices.
83 DS-4	MT	Look, you can see in the exercises; we can see after solving 1-2 exercises the meaning of what I made you write. Is it okay? [*meaning 'did you write'*]

Ayşe defines the basic mechanism of making the proof as getting one side from the other side, and she uses this approach frequently.

Examples:

ND-4: (16),(18),(19),(21)   ND-9: (268),(335),(352)

19 ND-4	MT	Look how we show commutative property. Matrices have commutative properties. We use one side of the equation and get the other side. When we are making a proof, we always use one side to get the other side.
268 ND-9	MT	By using one side, you will get the other side.

Although Ahmet gave limited time to proving in his geometry classes, almost all lessons (e.g. ND-2, ND-3 [lines 1-60], ND-12) in which proof is conducted, time is allocated in order to construct all parts of proving by interactive communication within the class, and by question-answer method.

Characteristic patterns of Ahmet's discourses depending on FD can be given as follows:

• Ahmet does not make proofs that depend on students' previous knowledge, or that are easy for the students;
• He allocates time for students to think about and work on the proofs;
• During this student work time, Ahmet walks around the classroom, checking notebooks and answers students' queries.

Examples:

ND-2: (3),(62),(64)   ND-3: (1)

3 ND-2	GT	We prove by drawing figures for each of them. Everybody tries by themselves. *[ST-2 is pointing out the question and asking something by whispering; and GT is going to the student to make some explanation and help him understand]*

Ahmet evaluates proofs conducted by students and the process of doing the proof in a formal way. He picks a student and makes them construct the proof step-by-step. He does not provide explanations about the proof from beginning to end (see ND-2: [2-45]).

Examples:

ND-2: (16),(65)   ND-3: (1)

Ahmet gives them enough time for making the proof, and does not go too fast. He helps students to comprehend the process of conducting a proof.

Examples:
ND-2: (2-45),(56-80)   ND-3: (1-23),(24-51)   ND-12: (8-30),(42-66)

Ahmet makes additional explanations to increase the understandability of the proof, both before starting and during construction of the proof. Similarly he makes explanations to the class through the student who is at the board.

Examples:

ND-2:   (67),(71)   ND-3:   (9),(24-27),(33)   ND-12: (1-7),(42-58)

67 ND-2	GT	First explain to your friends why, and then let's make the proof; but first you make the explanation.
9 ND-3	GT	Now let's look at the figure from the beginning and go through (look at) the explanations. AA' id perpendicular and so is D plane [*to ST-12*]. Does everybody follow that from the board?

He sometimes asks leading questions to the students when conducting the proof.

Examples:

ND-2:   (5),(7),(9),(71)   ND-3:   (3)

5 ND-2	GT	Is that place is a right-angled triangle?
9 ND-2	GT	Alright, do you use Pythagorean, or use similar triangle method? Suit yourself.

Ahmet stated the general mechanism in proving is the connection between hypothesis-conclusion or given-asked. Ahmet stated this relationship both through explanation and also by conducting mathematical representations; he defined proof by way of reaching conclusion from the hypothesis.

Examples:

ND-2:   (35),(52-55),(71-80)   ND-3:   (13),(35-30),(41),(54),(96)

53 ND-2	GT	What do you want to show after that? [*by using* $\left.\right\}\Rightarrow$ *sign in the writing on the board*]. To show; AB is longer than AA'
		(brd: $\left.\begin{array}{l} AA' \perp (D) \\ B \in (D) \end{array}\right\} \overset{?}{\Rightarrow} |AB| > |AA'|$) This is what I want to show. Now, when we draw BA', BAA' is a right triangle.
77 ND-2	ST-4 GT	Length of $A'B$ is equal to the length of $A'C$. I want to show this, okay?
		(brd: $$\left.\begin{array}{l} AA' \perp (D) \\ |AB| = |AC| \\ \quad\Downarrow \\ \quad a \end{array}\right\} \overset{?}{\Rightarrow} \begin{array}{l} |A'B| = |A'C| \\ \quad\Downarrow \\ \quad b \end{array}$$
78	GT	"a", which is on the left is called the hypothesis. "b", which is on the right is what we have to prove.

Lexical Items

The finding of lexical items connects with discourse patterns in the field of discourse for MT and GT, as given below.

When a discourse is investigated according to FD; it should answer two questions about lexical items.
   • To which field the words in the discourse belong? (Semantic field)
   • In the discourse, how well the lexis are known by the general audience and specialists? (Specialization)

For the first question, the researchers investigated the words in the ND for both the mathematics and geometry classes, and listed the words used towards proof by students and their teachers. To prepare this list, not only were the words assembled, but also the section of statement that involved those words. A long (five page) pre-list was prepared and then Table 1 was created by using focused words. When Table 1 is investigated, it can be seen that words like theorem, proof/evidence, and show that are used by both Ayşe and Ahmet. In addition, Ayşe uses the words "formula" and "induction", whilst Ahmet used "principle" (Principle of Cavelieri) once. In mathematics topics for the semester there is proof by induction, the use of induction was found extensively. Ayşe used the word theorem once. When students were instructed to write about the topic while doing the proof, Ayşe expressed theorem and said "say definition or theorem" (ND-1 [10]). Moreover, Ayşe used refuting (ND-9 [286]) once, when a student used modular arithmetic, which is wrong for a proof when asked to prove by induction.

Table 1. Words in teachers' discourses towards proof

#ND & Class	Word/Term (used by Teachers)	Statement Number in Transcription
ND-1 (math)	*Show that*	*(3)*
	Proof /Evidence	*(8),(21),(55)*
	Theorem	*(10)*
	Formula	*(20),(48),(59),(64)*
ND-4 (math)	*Show that*	*(15),(19)*
	Proof /Evidence	*(15),(19),(40),(51), (60)*
ND-5 (math)	*Formula*	*(1),(49),(50)*
	Proof	*(9),(11),(50)*
ND-6 (math)	*Proof*	*(25),(26)*
ND-8 (math)	*Show that*	*(2),(22)*
ND-9 (math)	*Proof*	*(189),(258),(282),(286),(297),(299),(308)*
	Show that	*(239),(241),(244),(274),(288),(304),(306),(313),(327),(335),(348)*
	Induction	*(189),(239),(241),(244),(297),(299),(303),(304),(306),(308),(313) (327),(350),(352),(354),(356)*
	Formula	*(310)*
	Refutation	*(286)*
ND-10 (math)	*Show that*	*(92),(94),(135),(151)*
	Proof	*(144),(146),(151)*
	Induction	*(144)*
	Formula	*(158),(160),(163)*
ND-11 (math)	*Proof*	*(75),(89),(111)*
	Show that	*(69),(89),(107),(114)*
	Induction	*(69),(71),(75),(81),(89),(107),(114),(116),(121)*
	Formula	*(38)*
ND-13 (math)	*Proof*	*(28),(36),(41),(43),(58)*
	Show that	*(45),(58)*
ND-2 (geo)	*Theorem*	*(1),(72)*
	Proof	*(2),(3),(45),(49),(51),(56),(67),(71),(78),(80)*
	Show that	*(74),(76),(77)*
ND-3 (geo)	*Theorem*	*(24),(25),(27),(52),(58),(95),(112),(120)*
	Proof	*(35)*
	Show that	*(39)*
ND-7 (geo)	*Theorem*	*(29)*
	Proof	*(35),(37)*
ND-12 (geo)	*Principle*	*(1),(3)*
	Proof	*(8),(30),(32)*

Table 2. Words in students' discourses towards proof

#ND & Class	Word/Term (used by Students)	Statement Number in Transcription
ND-1 (math)	*Proof /Evidence*	*(7),(54)*
	Induction	*(13)*
	Formula	*(35)*
ND-4 (math)	*Evidence*	*(39),(59)*
	Show that	*(17)*
ND-5 (math)	*Proof*	*(10)*
ND-6 (math)	--	--
ND-8 (math)	*Proof*	*(7)*
	Induction	*(5)*
ND-9 (math)	*Proof*	*(190),(298)*
	Show that	*(240),(285)(307),(309)*
	Induction	*(240),(285),(298),(307),(309),(330)*
	Formula	*(191),(309)*
ND-10 (math)	*Induction*	*(80)*
	Formula	*(13)*
ND-11 (math)	*Induction*	*(118)*
ND-13 (math)	*Proof*	*(35),(40)*
ND-2 (geo)	*Theorem*	*(12)*
	Proof	*(15),(18),(59)*
	Show that	*(83),(87)*
ND-3 (geo)	*Theorem*	*(8),(119)*
	Proof	*(6)*
ND-7 (geo)	--	--
ND-12 (geo)	--	--

Ahmet used the word theorem more often than Ayşe. When students' discourses are investigated, they used these words more cautiously than their teachers. Students most frequently used induction and show that among the words expressed in mathematics classes, and proof in geometry classes (see Table 2). When any high school or university mathematics/geometry textbook is examined, and a list the words created related to proof and proving, the following words would be likely; definition, proposition, theorem, axiom, postulate, assumption, lemma, mid theorem, final theorem, proof, presumption, conjecture, argument, hypothesis, sampling, verification, refutation, show that, deduction, and induction. Ayşe mostly used proof, induction and show that in her classes, and Ahmet used proof and theorem, and so the two most used words (proof, show that) in their discourses matched the list from the books. This data shows that in these communication situations, the repertoire towards proof and proving are very limited in the classroom. Naturally, it cannot, however, be assumed in general by only considering the words used by teachers.

Discourses towards proof and proving are affected by topics in both mathematics and geometry, with propositions, theorem and proofs present in the curriculum. However, only two or three words and the communication constructed around these words' semantic meanings can create a limited form of a general terminology in students' learning towards proof. In another words, presenting terminology with such a narrow perspective (with regard to diversity), without detailed explanation of the meanings of the words, and without using variety of contexts, could cause problems, not only for improving classroom terminology, but also for learning conceptually.

These kinds of evaluations also involve answers to the second question, which considered how well the general audience (students) and specialists in audience (teachers) understand the revealed words. PD provides important data to depict some findings towards the second question. Especially when PD-1 is investigated, the verbal discussions with Ayşe and her students about "assumption, axiom, theorem, and proof" helped to comprehend the students' conceptual understanding and associations towards mentioned terms and to see the problematic perspectives.

Table 3. Prompted discourse text (PD)-1A

(A) 9-33		April 28, Monday (Math-3) PD-1
10	MT	ST-5 come on, what is the statement? Tell us your thinking.
11	ST8	May I say Sir?
12	ST3	Those are sentences that declare definitely false or definitely true.
13	ST8	No, it can be true statement, false statement, to me they are command/ result sentences.
14	ST11	He [*pointing at ST-3*] means that!
15	MT	They report verdict, but what kinds of verdict are reported?
16	ST3	It is either definitely true or definitely false.
17	ST2	Definite result.
18	MT	It reports a results either definitely true or definitely false, alright ST-2 tell me, say a statement.
19	ST2	Tuesday is a weekday.
20	MT	Tuesday is a weekday.
21	ST7	It is a true statement.
22	MT	Yes, what else?
23	ST11	Every fan of Fenerbahçe team are sad.
24	MT	We don't generalize. Look we did not say generalization, we said statement.
25	ST8	Okay, what if we say today is Wednesday, is it a true or false statement?
26	MT	This is a false statement.
27	ST8	Today is Monday, but if it were Wednesday, I mean this statement is not definitely false or true.
28	MT	What if we say today is Monday, isn't it true?
29	ST8	What if I say this sentence tomorrow?
30	MT	Ha, if you say this in that day it would be false.
31	MT	Umm, ST-6 was saying a statement.
32	ST6	Ankara is the capital of Turkey.
33	MT	Yes, okay. What is theorem?

In the first section [10-32] of PD-1 (see Table-3), there is a discussion about what a statement is, with examples. In general, the class has similar ideas, and except for MT's answer [23-24] to the relationship between generalization and statement, it can be seen that there is no conflict on ideas. It may be because of an idea that statements do not involve generalization because of MT's "we do not generalize, look we did not say generalization, we said statement". After that, MT asked the definition of theorem and after ST-5's answer the discussion is changed towards axiom [37-80].

This second section (see Table-4) has a different discourse construction, in which different ideas and examples appear towards statements. Students are defining axiom as 'cannot be proven as true (or false), but can be accepted as true intuitively'. Although there is not a big problem about the definition, expressing his opinion to prove axioms, MT [48] deepens the discussion to reveal some misconceptions. ST-7 claimed that if axioms are proved, then it would be law [49] (incorrect) and ST-11 repeated the same wrong idea for theorems [58]. Therefore, some students brought perceptions about the concept of law from other science branches (e.g. Newton's law of motion) into mathematics; they perceive every provable thing can be a law.

When Ayşe asked for examples of axioms, ST-1 gave a point [54] and a line segment [59]; ST-7 gave the sum of the angles of a triangle as 180o [56]; and ST-8 gave the area of rectangle is a.b [61]. In the classroom there was no agreement on the examples as axioms or not, nor were they law or theorem. MT's claim of the area of a rectangle as a.b being provable was another provocative debate about the difference between accepting and proving [62] statements. However, when MT asked for a definitely true axiom, only ST-11 gave an example [80] of one of Euclid's well-known axiom. Moreover, in ST-11's explanation of the area formula of a rectangle [67, 69, 71], different examples are presented about perceived proof depending on previous learning. In PD-1 a discussion [81-132] on theorem takes places after the axiom debate. Students claimed that theorems cannot be proven, that they have some doubts (1-2%), but they help them to make correct computations (see Table-5).

Table 4. Prompted discourse text (PD)-1B

(B) 9-33		April 28, Monday (Math-3) PD-1
37	ST5	Axiom.
38	MT	Axiom, well, what is it?
39	ST7	"Belit Belit[†]"
40	MT	What is "belit"? [it was the first time MT heard it!]
41	ST9	Sir. We couldn't prove the trueness exactly…
42	ST3	But it couldn't be proved as falseness...
43	MT	One minute, one minute, please guys, raise your hand to speak, ST-9 said something.
44	ST9	Couldn't prove falseness or trueness definitely, but intuitively accepted.
45	MT	I mean, it is not 'couldn't prove'... let's say it appears a little truer.
46	ST9	The trueness of it can obviously be seen, that is trueness can be seen intuitively….
47	ST6	But it is not proven/ could not prove….
48	MT	Why it is not proven, it can be...
49	ST7	If we prove it, it would be *law*.
50	SS	It is not proven.
51	ST9	It cannot happen [*referring to it is not proven*]
52	MT	Okay, tell me one axiom then.
53	ST11	One minute, one minute, I want to say something.
54	ST1	Point, point.
55	MT	Is point an axiom? That is a definition.
56	ST7	Can I tell you? The sum of the angles is 180°.
57	MT	That is a theorem.
58	ST11	But sir, we said that if theorem is proved it would be a law, which is proved.
59	ST1	Line segment, line segment….
60	MT	One minute [*ST-8 is raising a hand*]. Yes, [*to ST-8*]
61	ST8	The area of the rectangle is a.b
62	MT	That is proved.
63	ST8	We couldn't prove.
64	ST2	It is not proved, but it is shown trueness or falseness.
65	ST11	It is proved, it is proved. Can I tell you?
66	MT	One minute, Okay, tell us ST-11.
67	ST11	Now, let's say there is a AB line segment with length a.
68	ST4	There are b-many from that.
69	ST11	As a pile, b-many from that.
70	MT	Ha, you put [*as a pile*]
71	ST11	That is a.b
72	ST4	We accept those, but.
73	ST2	Yes, that is.
74	ST11	Yes.
75	ST2	Yes, that is not proof, I mean.
76	ST9	Since we called 180° as a value [*value of an angle*] of the straight angle, it is 180°.
77	MT	Alright, let's say an axiom. Is there anyone who wants to say an axiom that can be obviously seen as true?
78	ST11	Can be obviously seen as true?
79	MT	I mean, you should say a very clever sentence, then...
80	ST11	From two points there is only one line passes.
81	MT	Yes, from two points there is only one line passes. Well, what is theorem?

[†] Belit is another meaning of postulate in Turkish.

Table 5. Prompted discourse text (PD)-1C

(C) 9-33		April 28, Monday (Math-3) PD-1
81	MT	Yes, from two points only one line passes. Alright, what is theorem?
82	ST11	Theorem involves numerical things...
83	ST5	Cannot be proved definitely, but can be intuitive.
84	ST2	That is "belit!"
85	SS	No, that's not, no.[to *ST-5's sentence*]
86	ST8	Sir! Sir!...[*he/she wants a word*]
87	MT	ST-7 tell us...
88	ST7	Now, for example, an idea should be proved with 100% precision to be a law or canon; in every experiment it gives the same result. In theorems, for example, if it is proved 98% or 99%, but if there is still some suspicions, even if just a little, it could be refuted. This is theorem.
89	MT	[*there is an intense, highly-charged discussion*] Alright, everybody is going to be able to tell us their ideas.
90	ST5	Is theorem and theory is same?
91	MT	Is theory and theorem the same thing? Everybody will tell us their ideas.
92	ST11	No Sir
93	MT	Do you agree?
94	ST11	I don't agree anything.
95	MT	Tell us.
96	ST11	Now, we are saying something 99% accepted, or could not be proven, but for instance we use Menelaus Theorem as accepting 100% true, and it gives same result every time. It is 100% true, but why we say it is not definite?
97	ST7	But Sir…
98	MT	[*to ST-7*] it is your theorem definition, I did not say anything.
99	ST7	But it is a law, but still there are hitches/problems.
100	MT	But what hitches/problems?
101	ST11	Dude, that theorem is not the same theorem.
102	ST7	That's what I want to say. To me they are used in science as concrete; in mathematics it is abstract, and because of this it is a theorem, a kind of paradigm [*in mathematics*]. I mean it is not definite.
103	MT	Let ST-1 speak. ST-1, what does the theorem mean for you?
104	ST-1	Not proved definitely.
105	MT	Theorem is not proved!!! Huh?
106	ST13	Not definitely...[*proved*]
107	ST-9	But we can prove theorem in the class.
108	ST-8	Cannot be proven as true, but it satisfies the data. What we found satisfies the data.
109	MT	Could we not prove the sum of the angles of a triangle as 180°?
110	ST11	We can prove, draw a circle.
111	ST8	We can prove that but, I mean Menelaus Theorem, [*but*] we can prove that one also.
112	MT	Really!
113	ST7	But we could not prove why a circle is 360°.
114	MT	Why we couldn't prove that?
115	ST7	Why, how we can prove it?
116	ST13	Why, it has been known for a long time.
117	ST7	I say 720°.
118	ST11	Even 450 at the beginning.
119	ST7	No, it was 400. [*discussion starts*]
120	MT	Don't speak at the same time. Tell us [to ST-6].
121	ST6	Sir, it was 400 and more, and it wasn't divided into anything, whereas 360 is good, can be divided by 3, 4, and 2 [*also*]
122	ST11	That's the only reason Sir. It can be divided into lots of numbers [*say so*].
123	MT	But I know the area of a circle can be proved.
124	ST11	It is not area, we said 360°.
125	MT	Is 360°? [*it*] is an agreement.
126	ST6	That is agreement.

127	ST7	I want to say something. They [*the people called this*] calculate 360°, but they first said the circle is 360°, then, according to this they constructed materials and then they measure with this material.
128	ST12	They also designated the angles.
129	ST7	Yes, those materials were already arranged according to a circle before.
130	ST2	Sir, what if the circle would be 400°. Is the sum of the angles of a circle 180°?
131	ST11	No it would be 200°.
132	ST9	It would be 200°.
133	MT	Okay, is statement and theorem related?

When the sentences of the students who understood theorem are listed in order, this approach can be seen quite easily (see Table-6). According to their sentences, their thoughts can be divided into two groups, theorems that have proof, and those that do not have proofs. That is, a proof that can be conducted fully, or one that has missing parts or sections.

Table 6. Students' theorem definitions

Line	Student	Sentence
82	ST11	Theorem involves numerical things...
83	ST5	Cannot be proved definitely, but can be intuitive.
84	ST2	Oh, that's "belit[‡]"!
85	SS	No, that's not, no.[to *ST-5's sentence*]
88	ST7	Now, for example an idea should be proved with 100% precision to be a law or canon; in every experiment it gives the same result. In theorems, for example, it is proved 98% or 99%, but if there are still some suspicions, even if just a little, it could be refuted. This is theorem.
104	ST1	Not proved definitely.
106	ST13	Not definitely...[*proved*]
108	ST8	Cannot be proven as true, but satisfies the data. What we found satisfies the data.

Again in this section, like in the discussion of theorem and law, there exists some diversity of views because of interrelationships between the same terms in other science branches - the discussion on whether theorem and theory is the same thing or not, and which is more definite, and has a proof is an example of that situation.

Table 7. Prompted discourse text (PD)-5

15-47		May 14, Wednesday (Math-1) PD-5
43	ST2	And also one thing Sir, we prove by accepting some specific things in mathematical proofs.
44	ST9	Some of the terms, Sir, to me they do not exist absolutely in mathematics.
45	MT	Why?
46	ST9	We create some terms by ourselves.
47	MT	Could you tell me, for example, what is created by us?
48	ST9	For example 1 is 1, because of us, we call it like that.
49	MT	How come?
50	ST9	I mean we said 1 to that number, so that is 1.
51	MT	Of course what else can we say? I couldn't understand that well.
52	ST9	Sir, we do create, make some terms…
53	MT	But it is accepted worldwide.
54	ST13	Is it in nature? No, not, for instance.
55	ST9	It is not in the nature, humans call it, and because of this it is not definite.
56	ST3	You called "bir" [*in Turkish*], an English called "one", in differently.
57	ST2	Sir, in most proof we call n=R (Real Numbers), but we can prove in restricted space.
58	MT	But is it a restricted space, real numbers are the biggest thing that we use besides complex numbers?
59	ST-2	In space there are lots of three dimensional numbers.
60	MT	But you can make proof even in the restricted space, right?
61	ST-7	But Sir, we can restrict the nature also.

[‡] Belit is another meaning of postulate in Turkish.

Students cannot be sure about whether the proven things are definitely true, full and provide convincing verification; because theorems are provable statements, but since something has to be accepted at first and then the proof is conducted, that makes the students doubtful. The statements accepted during the proof appeared different for the students than the acceptances (e.g. definitions, axioms) used by those capable of making proofs [72, 76, 125, 128]. Discourse to support this appeared during the discussion in PD-5 [43-61] (see Table-7).

Table 8. Prompted discourse text (PD)-1D

(D) 9-33		April 28, Monday (Math-3) PD-1
133	MT	All right, are statement and theorem related?
134	ST-9	Yes Sir. They are related because theorems are born from statements.
135	MT	Pardon, I couldn't quite understand.
136	ST-9	I mean Sir, because they are related.
137	MT	Related, why related?
138	ST-9	Theorems are born from the statements. A scientist puts out a statement and produces theorems while proving it.
139	MT	What else? [*an idea*] [*some of the students say they agree with ST-9*] Yes, it is a nice idea. Is there any opinion?
140	ST-7	To me, theorems are expressed by statements.
141	MT	Theorems.
142	ST-7	Expressed by statements.
143	MT	For instance, how are they expressed, say a theorem and I can see hypothesis and result.
144	ST-8	For instance Pythagorean theorem.
145	MT	Pythagorean theorem. But where, in a right-angled triangle?
146	ST-7	We say in a right-angled triangle, the addition of the squares of two sides is equal to the square of the <u>hypotenuse</u>.
147	MT	Alright, which part is the hypothesis of this theorem? [*3-4 minutes silence*]
148	ST-8	The equality sentence.
149	ST-2	The formula is hypothesis $a^2 = b^2 + c^2$.
150	MT	What is hypothesis? [*to the class*] Isn't it the premise?
151	SS	Yes.
152	MT	What is result? Isn't that what we the asking for?
153	SS	Yes.
154	MT	Alright, you said in a right-angled triangle you said Pythagorean theorem.
155	ST-8	Is $a^2 + b^2$ hypothesis, and c^2 the result?
156	MT	[*he shakes his hand to say no.*] What else?
157	ST-11	Can I say?
158	MT	Yes.
159	ST-11	ABC is a right-angled triangle. a,b are perpendicular sides and c is the <u>hypotenuse.</u>
160	MT	Is the premise "a right-angled triangle" a hypothesis, or is it a premise?
161	MT	If I say this is a right-angled triangle, opposite side of 90°, that is if I accept angle A is 90° then $a^2 = b^2 + c^2$ [*looking at ST-8.*]. What is hypothesis in a theorem, it is part premise, so what is the result? Alright, say another theorem and separate the hypothesis from the result.
162	ST-11	Okay, let's say.
163	MT	Say it.
164	ST-7	For example, cosine theorem.
165	MT	Cosine theorem.
166	ST-7	For example, a triangle, I guess, a triangle let's have a, b, c as sides.
167	MT	Yes.
168	ST-7	From that $a^2 = b_2 + c^2 - 2bc.cosA$
169	MT	Okay. What is proof then?

There are some deficient and restricted thoughts on existing mathematical objects, how mathematical knowledge is produced and, in the next procedure, how these objects and knowledge are used. Like in the example sentence "It is 180°, since we called 180° to a value [measure] of straight line" [76], for acceptance of mathematically explained concepts (verbal, symbolic or figurative) that involves a meaning or explanation that

is not guaranteed, or not true from a mathematical perspective, it is not going to be understandable to the students.

Table 9. Prompted discourse text (PD)-1E

(E) 9-33		April 28, Monday (Math-3) PD-1
169	MT	Alright, what is proof?
170	SS	Proving.
171	MT	Proving.
172	ST8	Use one side to reach the other side. [*Also ST-2 says same thing*]
173	ST11	But we will use only one side.
174	MT	What do you mean by saying use one side? Whether we use hypothesis?
175	SS	Hypothesis, hypothesis, result.
176	MT	Do we say proof when we get a result by using hypothesis?
177	SS	Yes.
178	MT	Alright, in this year we had a class about proof by induction in mathematics [*before*], didn't we? What was proof by induction, is there anyone who remembered [*they ask for permission to speak*]?
179	MT	ST-8 can tell us.
180	ST8	We show the trueness for k=1, then we accept the trueness for k=k.
181	SS	n, n should be equal. [*they corrected ST-8's words*].
182	ST8	We accepted trueness for n=k. then, we write n=k+1 in order to find k, if we see it is true then we say that it is true.
183	MT	Alright, from memory, an easy one. I will write one of four remembered. [*consults his notes*].
184	ST5	And also there is deduction.
185	ST8	It is in philosophy.
186	ST4	In mathematics is there deduction? [*MT consults his notes, classroom is a little noisy and the question is ignored*].

Since mathematical objects are abstract and a product of the human mind, the specific names of these objects and their properties are not fully accepted. Except for terms without definitions, axioms, and computational routines, the construction is revealed when definitions and rules are produced through logical inferences, and by interactive relations with horizontal and vertical connections. In a holistic perspective of proofs the point that students have difficulty with are things that are accepted to reduce the absoluteness and trueness of the proofs. There are more concrete examples to this situation in the discourses about proof by induction, in ND-9/10/11. In the following excerpt (PD-1 [133-169]), the relationship between statement and theorem is discussed (see Table-8).

When looking at students' sentences in response to questions from MT about relationships between statements and theorem, there are no divergent or opposing ideas. However, when MT asked them to separate the hypothesis and the resulting theorem, some students experienced problems [148,149,155]. Before PD-1, MT conducted proofs four times and GT three times. GT especially devoted either a large part or whole lessons to conducting proofs. Moreover, GT constantly stressed the mechanism of given/hypothesis with both verbal and mathematical representation during constructing of the proofs during question and answer sessions. On the other hand MT presented the mechanism of proof as realizing one side from the other side of the equation. However, based on the discourse data (e.g. Pythagorean Theorem) the teaching method depends on whether the information given directly by the teachers is insufficient for the students' understanding and applications of proof mechanism. The final section [169-186] of PD-1 (see Table-9) looks at what proof is, where perspectives are dependent on the MT's (the part in ND before PD-1) ideas and guidance is dominant. Proof is getting one side by using the other side of the equation. This opinion is the general perspective of the class on the basic mechanism of the proof. This mechanism is repeated frequently in ND by MT (e.g. ND-4: [16],[18],[19],[21], and ND-9: [268],[335],[352]).

In PD's it can be seen that the most important point on students' learning about proof is this basic mechanism. MT restates this mechanism by stated use of the hypothesis to obtain results. Similarly, in PD-5, MT stated that "of course in mathematics first there is a given statement and also a wanted statement, dependent on the former.

You use the givens and do the proof, right?..." [38]. In the discourses related to what is proof, MT makes students remember the rule of 'proof by induction', which is also a kind of proof [178].

Table 10. Prompted discourse text (PD)-2A

(A) 10-37		April 30, Wednesday (Math-1)PD-2
29	ST2	Deduction [*ST-7 said at the same time.*]
30	MT	What is deduction?
31	ST6	From general to specific.
32	MT	How? Give an example.
33	ST5	From general, backward.
34	ST3	I mean, we accept a thing before we investigate in detail.
35	ST7	Can be like this? For instance, we prove by deduction, hmm the thing?
36	MT	What?
37	ST7	For instance, for the sequence $1^2 + 2^2 + 3^2 + ...$, we said it is $n.(n + 1).(2n + 1)/6$
38	MT	Yes, yes…
39	ST7	For instance, $n.(n + 1).(2n + 1)/6$, so $1^2 + 2^2 + ...$ and so on and so forth.
40	MT	Can we find?
41	ST8	In philosophy we saw this, like this…
42	MT	What did you see?
43	ST8	[The] 11-science [class] is hardworking. ST-1 is one of the students of 11-Science. Then ST-1 is hardworking, we saw that this is deduction.
44	MT	You see, you discussed proof in philosophy. Alright, who can make proofs?

During the last part of the discussion on proof, there was a question about whether or not in mathematics, deduction has a place. This question, stated here and also in PD-2 [29-44], is discussed extensively. This question is proposed both here and also in PD-2 extensively (see Table-10).

Table 11. Prompted discourse text (PD)-2B

(B) 10-37		April 30, Wednesday (Math-1) PD-2
1	MT	What are the proof methods?
2	ST11	*We accept one side and obtain the other side.*
3	ST2	Induction.
4	ST8	There is direct proof, and indirect proof, so it is divided into two.
5	ST11	Yes.
6	ST8	There was also proof by contradiction.
7	MT	Say it again.
8	ST8	There is also proof by contraposition.
9	MT	The method proof by contraposition [*meaning approving*]; look it is very good, as ST-11 said using the left part to get the right part.
10	ST11	Or by using the right part.
11	MT	*What was the proof method on getting the left part ST-8, let's hear it.*
12	ST8	It was divided into two; proof by contraposition and also direct proof.
13	ST2	Sir. Was proof by contraposition a confirmation?
14	ST8	And also direct proofs are divided…No, there is proof by contradiction, and then there is proof by contraposition.
15	MT	ST-2 said something.
16	ST2	Is proof by contraposition like a confirmation?
17	MT	*Proof by contraposition isn't like confirmation; you are giving an opposite example to show trueness.*
18	ST11	But it can be falseness also.
19	MT	Or you show falseness. Okay, what else do we prove by induction?
20	ST11	It can be also induction, can I say something?
21	MT	Yes.
22	ST11	For example, it is not right for all, that x does not verify the value, we show it definitely does not verify, this is a proof, isn't it? The opposite one is true.
23	ST7	He/she means the falseness.
24	MT	That can be, why not.
25	ST11	But it cannot be, for all x's may not verify.

(B) 10-37		April 30, Wednesday (Math-1) PD-2
26	MT	It may not verify, what else, is there anyone?
27	ST11	We can do induction.
28	MT	Yes we're using induction this year.

Discourses on whether or not deduction is a method of proof suggest that the problems of students' learning and comprehension of general concepts of proofs are an important finding of the study. It showed students' knowledge to be ambiguous. Since deduction is discussed by using explanations and approaches from philosophy and could not give mathematical exemplification, the knowledge of the students on proof with deduction was found to be ambiguous. It is possible to say that similar is true for proof methods (see Table 11: PD-2, [1-28]).

Mathematically, deductive proof is a more acceptable proof than induction, and is generally known as a rigorous proof by mathematicians. Deductive proofs are reported in the mathematics curriculum (MoNE, 2010, p. 84) as direct and indirect proofs (indirect proof methods listed as; proof by contradiction, proof by contraposition, trial method, and proof by constructions/ proof by cases). It's a basic proof method that involves whole proof methods, except induction. According to discourses on proof methods [1-28] in PD-2, it's a basic method, with students discussing whether or not deductive proofs exist in mathematics; yet, as no examples are given, there may be deficiencies with both teaching approaches on proving (method of presentation) and students' comprehension.

Other findings in PD-2 [1-26] are; emphasis of the MT about obtaining one side from the other in equations [9,11], and the dialog (ST-2, MT) about what is proof by contraposition [13-19]. Discourse between MT and ST-2 involves different things about proof by construction, i.e. giving an opposite example method. While MT couldn't explain or provide any example of this method, she stated "proof by contraposition is not like confirmation, you are giving an opposite example to show trueness" [17]; ST-2 stated doubts about whether or not this method is confirmation. There were also discussions about proof mechanism in the relationship between proof and confirmation, and the meaning of the words used for proofs in PD-3 (see Table-12). In mathematics, when MT was asked what can be proven, instead of giving answers such as statements and theorems according to the literature, MT gave some general explanations of proof mechanism. Other statements [25-31] show perceptions about what proof is, or what kinds of function there are.

Table 12. Prompted discourse text (PD)-3

12-41		May 7, Wednesday (Math-1) PD-3
19	MT	Alright, in mathematics what do we prove?
20	ST-7	Actually we prove every possible thing.
21	ST-5	Everything that is possible to prove!
22	ST-3	*If there is left and right, we can absolutely prove it*
23	MT	What do you mean left and right?
24	ST-3	I mean we get the one part by using the other part.
25	MT	In short, you learned that you do not do confirmation. You are just using one side to get other side.
26	ST-9	Sir. Isn't confirmation a proof?
27	MT	Can proof be a confirmation?
28	ST-11	Proof is a confirmation.
29	MT	Yes.
30	ST-9	How can I know? We verify things; if founded and verified, does that means it's true?
31	ST-11	But all x confirms? "all" is important there.

Although students comprehend the general idea of mechanism presented by MT, when PD's are investigated, they have insufficient comprehension and ability about how the mechanism works, is constructed, and how it can be evaluated. Students (ST-2, ST-9) relate the proof and confirmation since they thought confirmation is a method applied to show the trueness of statements. Proofs are mathematical constructions referenced to show things (mathematically, generally expressed as formulas) are true. However, confirmation by verification, as a basic function of proof, has many different meanings. The reason that students could not see the difference depends on the presentation of a restricted perspective on the conceptual meaning of proof.

In summary, according to the findings of the process of discourses between students and teachers on poof and proving, they did not use words correctly or efficiently, neither terminologically nor conceptually. Students do not construct the meaning of the terms correctly in their vocabulary about proof and proving when students' definitions, explanations and exemplification are investigated in detail.

Discussion

Vygotsky's (1962) social constructivist theory characterizes learning as a social activity and emphasizes learning as active student engagement. Vygotsky also emphasizes the importance of speech and communication (discussion), social interaction (generalizing and higher order thinking) and most importantly, the connection between thought and language (development of higher-order concepts) about the language of mathematics. The investigation of the language of mathematics discourse analysis is the basic tool for researchers. In this current study, classroom discourse between student and teacher is examined through the proof and proving context.

When characteristic patterns of language used by teachers on communicational structure involving proof and proving are investigated, the context of the discourses consist of mathematical topics and interpretations of proof, with any significant differences between teachers towards these interpretations identified. When situations involved in proofs in the math classroom are considered, Ayşe applied lecture as the teaching method, giving information through direction and orders on proof to the students rather than discussions or questions-answer methods. Conversely, Ahmet's teaching methods were dependent on classroom discussions and question-answer methods. Therefore, like their teaching methods, discourses on characteristics of patterns also differed. The difference between the teachers' nature of teaching may cause this dissimilarity of the characteristics in their discourses at some points. The discourse flows from teacher to students when the teacher accepts a lecturing method with a very directive way; on the other side, it flows among the students, or from students to teacher, when the teacher tries to conduct more (partially or totally) student-centred classes. Ahmet prefers scaffolding perspective, helping students to create meaning from proofs. He gives time to students to think about proofs, encourages them to construct their own proof, and leads them to write the proof. At the same time, he encourages students to think about proofs and to follow the process in order to improve the level of comprehension of the proofs.

A study investigating teachers' discourses (Atweh et al., 1998) asserted that it was possible to have differences between two different teachers teaching same course with the same material. The researches stated that teachers with different genders and from a variety of socioeconomic groups had different discourses in the classroom. Moreover, Back (2005) stated that if students were exposed to highly mathematical talk, but not given the chance to participate, they were unlikely to develop mathematical form of life. But, if students engage in mathematical discussion freely, and express their thoughts and ideas with socially open mathematics registers, then they may generate mathematical form. In the light of this, it might be said that Ahmet's students had a chance to improve their mathematical life and registers where he encouraged them to improve the amount of mathematical communication (compared to other teacher's classroom). In the classroom discourse examined by Thornton and Reynolds (2006), the teacher organized communication and directed students in their thinking. It was also observed that discourses were not about the material process that were a part of the problem-solving procedure in traditional classrooms, but about mental processes. Additionally, in the study, it was observed that there are some dominant statements used in classroom dialogues such as 'in my opinion/up to me', which could produce general classroom discussions, helping students to criticize both themselves and other students' thoughts. The structure of the classroom of the teacher was not considered to be a reproductive form, which was usual for a traditional classroom, but to be a generative one, similar with the current study's classroom. In the current study, it was observed that the discourse in the Ayşe's classroom was reproductive where her students were mostly confirmative and fulfilling the instructions, but Ahmet presented a generative form of discourse in a certain extent where his students were led to experience reasoning processes.

In a research by Shreyar, Zolkower, and Pérez (2010), when teachers direct classroom conversation through lexico-grammatical choices, student participation in the conversation is realized. However, Herbel-Eisenmann and Otten (2011) report that in the observed classrooms, although the amount of speech is different for each classroom, it is still at a low level. Moreover, the timing of the speech of students also differs. In one of the classrooms, students give short answers, yet in another classroom students give long and detailed answers, complete with explanations and justifications. The researchers assert that using long verbal explanations and frequent speaking make for more effective mathematical learning. It is also observed that in Ahmet's classroom students have longer discourses depending on explanations. Herbel-Eisenmann and Otten (2011) state that discourses involving longer and more frequent answers by students influence their effective mathematics

learning. Because of this, it can be said that the discourse construction of Ahmet's classes have relatively more potential to effect students' learning of proof and proving. However, this is not that easy to accomplish since Schleppegrell (2007) claims that supporting students' competences with mathematical conversation takes time and serious thought.

In the current study, it is observed that some words (theorem, proof and show that) are used by both teachers, and some words with differing frequencies. Although there may be more words in ordinary mathematics and geometry classes expressing proof (from textbooks or literature), the teachers vocabulary is extremely limited, both in number and variety. The reasons for this limitation may be related with not giving enough importance to process and mechanism of constructing proofs and the provided learning environment depending on solving exercises/problems rather than conducting mathematical inquires. Similarly, students' vocabulary is also very limited, mostly using "induction and show that" (mathematics) and "proof" (geometry). So, the reason may be related with the teachers' limited usage of words because teachers are the main characters, who lead the communication processes. This verifies that in-class communication about proof and proving is limited with both teacher and students' restricted vocabulary. Although using fewer mathematical terms does not necessarily make for a poor mathematical register (the meanings, words, and structures used to express mathematics). According to Schleppegrell (2007), the register does not just include written and spoken language; she points to mathematics using a variety of semiotic systems in knowledge construction: "symbols, oral language, written language and visual representations such as graphs and diagrams" (p. 141). In this sense, the current classroom has a poor mathematical register due to inadequate usage of mathematical symbols and oral-written language. Communication constructed in the classroom around just one or two words and their semantic meanings create limitations on producing terminologies for topics, which then affect student understanding (about proof and proving). Using limited words during in-class communication can also be due to discourse characteristics, e.g. Ayşe's situation provides less discussion opportunity as she mostly dictates and students follow orders. Teaching methods dependent on lectures and information given directly to students does not facilitate students' understanding. Without explaining the meanings of words, or giving examples and varieties of usage, may hinder improving classroom terminology and conceptual understanding of proof and proving. Aligned with this, Wakefield (2000) claims that when teachers model correct mathematical speech, and use mathematics language in the classroom, students will eventually do likewise. Herbel-Eisenmann et al. (2014) reveal that teachers' discourses about mathematical registers and generating meaning shifted over time, from simple mathematical vocabulary to realizing differences through usage of mathematical words from everyday life, as well as the use of verbs and idioms according to researchers' practice. This shows that some processes conducted to teachers could positively improve the classroom discourse. In another study conducted on improving teachers' awareness, Freitas and Zolkower (2011) reveal that lesson study activities affect using semiotic tools more efficiently in the classroom, and also teachers' awareness improves through identifying the differences among the language of mathematical register and daily life. Therefore, teachers should use terminological language in their classes to improve students' language about the mathematical register. Temple and Doerr (2012) suggest that when teacher interactional strategies focus or probe student thinking, they are effective in supporting students' learning of the mathematical register and engage them in constructing new mathematical understanding. When teachers' discourses develop with qualitatively and quantitatively, their students change in the same direction.

Along with using few words for proof and proving, emphasis of the teachers on the aim and functions of proofs also differs. Simpson (1995) differentiates between "proof through logic" (the formal way of proof), and "proof through reasoning" that involves investigation. In the current study, one teacher emphasized practical applications and computational rituals, whilst the other emphasize connections of premises-conclusion or given-warrant. This distinction may come from accepting different perspectives on definitions/functions of the proofs. Hanna et al. (2009) state that proof has different meanings and functions in today's professional mathematics. For example, Nolt, Rohatyn, and Varzi (1998) define proof as deriving results from hypotheses using generally valid rules of inference, whereas Bundy, Jamnik, and Fugard (2005) refer to Hilbert's definition of proof as "a sequence of formulae each of which is either an axiom or follows from earlier formulae by a rule of inference". De Villiers (1990) explains five functions of proofs as "verification, explanation, systematisation, discovery and communication", on the other hand Hemmi (2010) classifies functions of proof in seven categories as "conviction, explanation, communication, aesthetic, systematisation, intellectual challenge and transfer". Moreover, Hemmi (2010) combines functions of proof and teachers' teaching strategies, and produces a theoretical model involving three styles of teaching: progressive style (I don't want to foist the proof on them), deductive style (It's high time for students to see real mathematics), and classical style (I can't help giving some nice proofs). Among the participants in the current study, Ahmet is using deductive style since he gives importance to couple of given-warrant, make students use especially the premises, and makes them use the critical thinking procedures. On the other hand, Ayşe is emphasises the verification function of the proof by

bringing practical usage of proofs into the forefront. Moreover, she uses progressive style since she sees proof as a tool for generalization after specific examples.

Besides different definitions of theorem, there can be misconceptions about its certainty. For example, some students thought that theorems were not proved by 100% precision. Moreover, differences between law, theory, and theorem were unclear, with some students having the idea that 100% proven theorems become law, confusing terms from other disciplines. According to students, differences among theory, theorem, and law depended on the exactness of the axioms; expressing unsureness about theorem having a definitive result or not. Sometimes, they were confused about the accepted statements (for instance axioms) that mathematicians used before starting the proof. Another confusion was about comprehension of the types of proofs. In the PD's, there were discussions about the kinds of proofs in mathematics. Students discussed whether or not deduction was a mathematical proof, or from other discipline (e.g. philosophy), presenting evidence of ambiguity of knowledge regarding types of proof. Although deduction is accepted as a rigorous proof in theoretical mathematics, students are confused if it applies to mathematics or philosophy. This confusion and discourses about comprehension of proof methods show deficiencies in students' knowledge about proof methods and issues with teaching methods. Similarly, Moore (1994) examined cognitive difficulties of university students in learning to develop formal proofs. He identified three major difficulties: conceptual understanding, mathematical language and notations, and getting started on a proof. Healy and Hoyles (1998) highlighted curriculum organization about proofs and classroom practices of students and teachers. Among the observations of our study, it was noted that teachers could not provide clear-cut explanations for students confused about proof methods, resulting in students remaining just as confused. Students' confusion was mostly about confirmation and theorem, defining confirmation as showing something is true. This definition relates to theorem, which also defines how to show things (axioms) are true. They could not separate the differences due to deficiencies in conceptual knowledge about proof.

Another confusion relates to definition of axioms: Is generalization an axiom? One teacher gives a wrong explanation to students, making them even more confused. Knuth (2002), and Lowenthal and Eisenberg (1992) observed that some teachers suffer the same inabilities and misconceptions as students. Teachers did not understand or were unconfident about understanding generality of proven statements and could not reliably distinguish between valid and invalid arguments (Knuth, 2002). Proof terms and methods of proofs were unclear in students' minds, with specialization of lexical items poor for both teachers and students. Lexical specialization of discourses for both teachers and students were not at an appropriate level. Some students said proofs were definitely true and complete; however, some said proofs did not have to be complete in order to be true, and might have missing parts or aspects. Classroom language (and the context involved in it) is important; a crucial component of the teaching-learning process since it can help to construct routines and regularities within which learning occurs (Voigt, 1985/1995). For instance, our findings revealed that students discussed the existence of mathematical objects and how mathematical knowledge is produced, but in this discussion they used some statements that is not based on a meaning in mathematics (e.g., a circle's 360° and in the definitions of axioms). Therefore we examined teachers' capabilities of orchestrating classroom discussions as we found students trying to understand how to construct mathematical knowledge, but their teacher could not adequately lead the discussion to produce a productive discussion environment. According to Ball (1993), one critical challenge facing mathematics teachers is orchestrating whole class discussions to advance mathematical learning.

In summary, neither teachers nor students had efficient DA or used vocabulary effectively, both terminologically and conceptually for their in-classroom communication about proof and proving. In the discourses, students especially could not construct the terms' definitions, explanations and exemplifications with their vocabulary. Herbel-Eisenmann et al. (2014) found teachers' primary focus about mathematics register was mathematics vocabulary, expecting memorization of mathematical terms and vocabulary; negating the meaning behind the vocabulary. Tackling this deficiency, they suggested engagement with reading and analysing students' work, and developing teachers' understanding to talk about mathematics register.

Results

Categorization regarding FA shows both teachers used limited numbers of words on proof and proving. Ayşe used mostly proof, show that and induction, whilst Ahmet used proof and theorem. Explanations of these words and exemplifications were very limited, plus discussions on meanings also relied on these inadequate terms. Students' discourses showed mostly just induction (mathematics) and proof (geometry). Especially in Ayşe's

classes, her passion to answer quickly resulted in misunderstanding the question; giving answers without thinking enough, and not giving detailed explanations also caused misunderstandings by students.

Both teachers and students employed very limited vocabulary on proof in classroom discourse; not only limited by numbers of words, but also limited by meaning appointed to these words and roles in the communication. Terminological meanings of words about proof insufficiently emphasize while conducting a proof; for instance, Ayşe hardly uses the word proof, and neither teacher states which definitions, axioms, or other theorems are used while conducting a proof (e.g. different meaning of proof in science, legal system, and criminology, and their differences with mathematics).

The meaning of proving something in mathematics requires verification, which uses concepts like sampling, generalization, confirmation, and abstracting; or using geometrical proof for algebraic proof. This requires careful usage of words to improve the conceptual meaning of proofs in different contexts (e.g. talking in daily life, not constructing connections to other disciplines, and not using resemblances).

Therefore parallel to these results, students' numbers of concepts and the lexical structures involved in their vocabulary also becomes limited. In her lessons, Ayşe allocates limited time for proof and insufficient time for students to construct their own perspectives towards proofs, and no opportunity for discussion. However, Ahmet allocates ample time, sometimes the whole lesson, to conducting proofs, providing opportunities for students to construct their own perspectives. However, when conducting the proof, dialog involves questions-answers, and whilst Ahmet answers questions, sometimes he dictates the approach to his students. According to DA's findings, teachers' discourses on proof and proving showed some differences regarding structure of discourse characteristics and how they were compiled. However, discourses didn't contain major differences in the process of learning and formulating meaning. One reason maybe that mathematics and geometry teachers embrace traditional teaching methods.

Communication was mostly teacher-centered, with students' learning on proof limited by the teachers' lectures and classroom questions and answers. In the process of constructing knowledge about proof and proving, the most efficient factor is teacher discourse. Teachers use lecture and question-answer methods, yet rarely conduct small group activities, don't set performance homework or projects involving proofs, don't use alternative proof strategies (e.g., visual proofs, two paragraph proofs, flowchart proofs, technology supported experimental proofs, proof comprehension tests) nor are proof problems set as either written or oral evaluations (except induction). Consequently, discourses in DAs involve one-way informational flows between teachers and students, with a structure that has no variety.

From investigating 18 lessons involving discourses about proofs, students showed positive attitudes and opinions towards proof, with proofs seen as both helpful and necessary. However, students are not clear about what is a valid proof, and lack comprehension about the place of proof in mathematics education - they only have knowledge about confirmation function of proof. Teachers' discourse directly affects students' cognitive structures, and the structure, variety and characteristic pattern of discourse are important components of this effect. The general finding is that the teachers didn't provide an appropriate environment for students based on the discursive components for communication of proof and proving.

Recommendations

Teachers raised awareness that classroom communication not only relates to lecturing, questions, problems-solving, and discussions with students that involve limited questions-answers and simple writing on the board. Especially nowadays, increasing the number of meanings and functions assigned to communication, the place and role of communication, both in daily life and teaching-learning process, requires better understanding. Communication is a basic target ability of mathematics curriculum and standards; an ability that enables teachers to perform different duties and responsibilities. Therefore, studies should focus on improving awareness of structure, context, kinds, and effects on students' understanding of teachers' discourses.

Mathematics teaching-learning processes, as the new mathematics curriculum suggests, should provide opportunities for students to be active, participative and open to communication, using both effective colloquial and mathematical language. This current study identified thoughts and opinion of students about proof and proving, and revealed discourses in teachers' ND's inadequate for identifying students' deficiencies, or the ways to improve them. Causality is linked to students limited participation in ND's, e.g. classrooms where teachers only follow traditional teaching methodologies, where it may difficult for students' discourses to be

appropriately structured, profound, and varied regarding proof and proving. Since students participating in lessons are mostly focused on application and questions problem-solving processes, it may not be possible to analyze sufficiently deep enough within the scope of discourses about students' perceptions on the level of conceptual understanding, the understanding and associating mathematical knowledge with existing misconceptions.

However, verbal mathematical discussions conducted among students and between teacher and students (e.g. PD-6) conducting proofing activity together can improve the DA's success and validity. Within the new mathematics curriculum philosophy, constructing the learning environment, planning the learning processes, and making students actively participate in the process will help to improve the teaching of proof. To overcome students existing deficiencies and limitations of knowledge about proof and proving, and developing proving abilities, one strategy to follow is increased opportunities to work with proofs, individually or in groups. In such activities, class discussions conducted by students and teachers during or after the proving process improve effectiveness, as do classroom learning processes where arguments, logical steps, and foreknowledge presented by teachers while producing and explaining a proof are made positively.

In DA studies about proof or other topics, it's helpful to use not only ND, but also PD and verbal explanations to analyze findings more deeply; the current study being an example. In proof-related research, while using PD's, one alternative method (e.g. PD-6) is where one or more proof problems are tackled by the whole class. Such applications help identify the abilities of proving, problems while conducting proofs, attitudes and beliefs on proofs, and from the research on proof schemas, multi-dimensional data makes it possible to reach more qualified findings and results..

References

Atweh, B., Bleicher, R. E., & Cooper, T. J. (1998). The Construction of the Social Context of Mathematics Classrooms: A Sociolinguistics Analysis. *Journal for Research in Mathematics Education*, *29*(1), 63-82.

Back, J. (2005).Talking to each other: Pupils and teachers in primary mathematics classrooms. In D. Hewitt (Ed.) *Proceedings of the British Society for Research into Learning Mathematics, 25*(2), (pp. 799-810).

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 94*(4), 373-397.

Benson, J. D. & Greaves, W. S. (1981). Field of Discourse: Theory and Application. *Applied Linguistics, II*(1), 45-55.

Bilgin, N. (2000). *İçerik Analizi*. İzmir: Ege Üniversitesi Edebiyat Fakültesi Yayınları.

Bundy, A., Jamnik, M., & Fugard, A. (2005). The nature of mathematical proof. *Philosophical Transactions of Royal Society, 363*(1835), 2377-2391. doi: 10.1098/rsta.2005.1651.

Cazden, C. B. & Beck, S. W. (2003). Classroom Discourse. In A. C. Graesser, M, A, Gernsbacher, & S. R. Goldman (Eds.), *Handbook of Discourse Processes*. New Jersey: Lawrence Erlbaum Associates. Inc. Publication.

Celce-Murcia, M. & Olshtain, E. (2000). *Discourse and Context in Language Teaching*. New York: Cambridge University Press.

Cem, A. (2005). Dilbilgisi Öğretiminde Biçim-Anlam-Kullanım Üçlüsü: Ders Malzemeleri Hazırlama ve Uygulama Önerisi. *TÖMER Dil Dergisi, 128*, 7-28.

Chapman, A. (1993). Language and learning in school mathematics: a social semiotic perspective. *Issues in Educational Research, 3*(1), 35-46.

Chapman, A. (2003). *Language Practices in School Mathematics, A Social Semiotics Approach*. The Edwin Mellen Press.

Common Core State Standards Initiative (2010). *Mission statement*. http://www.corestandards.org/

De Villiers, M.D. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Eggins, S. (2004). *An Introduction to Systemic Functional Linguistics* (2nd Edition). London: Continuum.

Ellerton, N. F. & Clarkson, P. C. (1996). Language Factors in Mathematics Teaching. In A. J. Bishop et al. (Eds.) *International Handbook of Mathematics Education*. Netherlands: Kluwer Academic Publishers.

Erton, İ. (2000). Contributions of Discourse Analysis to Language Teaching. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 19*, 201-211.

Fairclough, N. (1992). *Discourse and Social Change*. Cambridge: Polity Press

Fawcett, R. (2000). *A Theory of Syntax for Systemic Functional Linguistics*. John Benjamin Publishing.

Freitas, E., & Zolkower, B. (2011). Developing teacher capacity to explore non-routine problems through a focus on the social semiotics of mathematics classroom discourse. *Research in Mathematics Education, 13*(3), 229-247.

Günay, V. D. (2004). *Dil ve İletişim*. İstanbul: Multilingual.

Halliday, M. A. K., & Martin, J. R. (1993).*Writing Science: Literacy and Discursive Power.* London: Falmer.

Hanna, G., de Villiers, M., Arzarello, F., Dreyfus, T., Durand-Guerrier, V., Jahnke, H. N.,…Yevdokimov, O. (2009). Discussion Document. In F. Lin, F. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the 19th International Commission on Mathematical Instruction: Proof and Proving in Mathematics Education (vol. 1).* National Taiwan Normal University, Taipei, Taiwan: ICMI Study Series 19, Springer.

Har, Y. B. (2007). The Singapore Mathematics Curriculum and Mathematical Communication. *Proceedings of APEC-TSUKUBA International Conference III, Innovation of Classroom Teaching and Learning through Lesson Study, Focusing on Mathematical Communication*. Tokyo and Kanazawa, Japan.

He, A. W. (2002). Discourse Analysis. In M. Aronoff & J. Rees-Miller (Eds.), *The Handbook of Linguistics.* Oxford: Blackwell Publishing.

Healy, L. & Hoyles, C. (1998). *Justifying and proving in school mathematics*. London: Institute of Education, University of London.

Hemmi, K. (2010). Three styles characteristing mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics, 75*(3), 271-291.

Herbel-Eisenmann, B., Johnson, K. R., Otten, S., Crillo, M., & Steele, M. C. (2014). Mapping talk about the mathematics register in a secondary mathematics teacher study group, *Journal of Mathematical Behavior, 4*, 29-42.

Herbel-Eisenmann, B. & Otten, S. (2011) Mapping mathematics in classroom discourse. *Journal for Research in Mathematics Education, 42*(5), 451-485.

Huang, J. & Normandia, B. (2007). Learning The Language of Mathematics: A Study of Student Writing. *International Journal of Applied Linguistics, 17*(3), 294-318.

Huang, J., Normandia, B. & Greer, S. (2005). Communicating Mathematically: Comparison of Knowledge Structures in Teacher and Student Discourse in a Secondary Math Classroom. *Communicating Education, 54*(1), 34-51.

Kharing, T. T., Hamaguchi, K., & Ohtani, M. (2007). Development Mathematical Communication in the Classroom. *Proceedings of APEC-TSUKUBA International Conference III, Innovation of Classroom Teaching and Learning through Lesson Study, Focusing on Mathematical Communication.* Tokyo and Kanazawa, Japan.

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*(1), 61-88.

Leech, G. (1983). *Principles of Pragmatics.* London and New York: Longman.

Lin, C. H., Shann, W. C., & Lin, S. C. (2007). Reflection on Mathematical Communication from Taiwan Math Curriculum Guideline and PISA 2003. *Proceedings of APEC-TSUKUBA International Conference III, Innovation of Classroom Teaching and Learning through Lesson Study, "Focusing on Mathematical Communication.* Tokyo and Kanazawa, Japan.

Lowenthal, F. & Eisenberg, T. (1992). Mathematical induction in school: An illusion of rigor. *School Science and Mathematics, 92*(5),233-238.

Marks, G. & Mousley, J. (1990). Mathematics Education and Genre: Dare We Make the Process Writing Mistake Again*? Language and Education, 4*(2), 117-136.

McCarthy, M. (1991). *Discourse Analysis for Language Teachers.* Cambridge University Press.

MEB (MoNE), (2010). Ortaöğretim Geometri Dersi 11. Sınıf Öğretim Programı. Retrieved from http://ttkb.meb.gov.tr/www/ogretim-programlari/icerik/72

MEB (MoNE), (2013). Ortaöğretim Matematik Dersi Öğretim Programı. Retrieved from http://ttkb.meb.gov.tr/www/ogretim-programlari/icerik/72.

Mechura, M. B. (2005). A Practical Guide for Functional Text Analysis: Analyzing English Texts for Field, Mode, Tenor and Communicative Effectiveness. Retrieved from http://www.cainteoir.com/cainteoir_files/etc/FunctionalTextAnalysis.pdf.

Miyagui, M. (2007). Key Questions for Focusing on Mathematical Communication. *Proceedings of APEC-TSUKUBA International Conference III, Innovation of Classroom Teaching and Learning through Lesson Study, "Focusing on Mathematical Communication.* Tokyo and Kanazawa, Japan.

Moore, R. C. (1994). Making the transition to formal proof, *Educational Studies in Mathematics, 27*(3), 249-266.

Morgan, C. (2006). What Does Social Semiotics Have to Offer Mathematics Education Research? *Educational Studies in Mathematics, 61*(1/2), 219-245.

NCTM, (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Nolt, J., Rohatyn, D., & Varzi, A. (1998). *Schaum's outline of theory and problems of Logic (2nd ed.)*. New York: McGraw-Hill.

O'Halloran, K. L. (2000). Classroom Discourse in Mathematics: A Multisemiotic Analysis. *Linguistics and Education, 10*(3), 359-388.

O'Halloran, K. L. (2004). Discourse in Secondary School Mathematics Classroom According to Social Class and Gender. In J. Foley (Ed.), *Language, Education and Discourse*, Continuum.

Radford, L., Schubring, G., & Seeger, F. (Eds.) (2008). *Semiotics in mathematics Education: Epistemology, History, Classroom, and Culture*. Rotterdam: Sense.

Renkema, J. (2004). *Introduction to Discourse Studies*. John Benjamins Publishing.

Rogers, R., Malancharuvil-Berkes, E., Mosley, M., Hui, D., & Joseph, G. O, (2005). Critical Discourse Analysis in Education: A Review of the Literature. *Review of Educational Research, 75(3)*, 365-416.

Ryve, A. (2004). Can Collaborative Concept Mapping Create Mathematically Productive Discourses? *Educational Studies in Mathematics, 26*, 157-177.

Schiffrin, D. (1994). Approaches to Discourse, Oxford: Blackwell.

Schleppegrell, M. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading and Writing Quarterly*, *23*, 139-159.

Setati, M. (2005). Mathematics Education and Language: Policy, Research and Practice in Multilingual South Africa. In R. Vithal, J. Adler, & C. Keitel (Eds.), *Researching Mathematics Education in South Africa*. Cape Town: HSRC Press.

Shreyar, S., Zolkower, B., & Perez, S. (2010). Thinking aloud together: A teacher's semiotic mediation of a whole-class conversation about percents. *Educational Studies in Mathematics, 73*(1), 21-53.

Sierpinska, A. (2005). Discoursing Mathematics Away. In J. Kilpatrick, O. Skovsmose, & C. Hoyles (Eds.), Meaning in Mathematics Education (pp. 205-230). Dortrecht: Kluwer Academic Publishers.

Simpson, A. (1995). Developing a proving attitude. *Conference proceedings: Justifying and proving in school mathematics*, 39-46. London: Institute of Education, University of London.

Steele, D. F., (2001). Using Sociocultural Theory to Teach Mathematics: A Vygotskian Perspective. *School Science and Mathematics, 101*(8), 404-416.

Straber, R. (2004). Introduction: Semiotics in Mathematics Education Research. *ZDM, 36*(6), 184.

Temple, C., & Doerr, H. (2012). Developing fluency in the mathematical register through conversation in a tenth-grade classroom. *Educational Studies in Mathematics, 81*(3), 287-306.

Thornton S. & Reynolds, N. (2006). Analysing Classroom Interactions Using Critical Discourse Analysis. In J. Novotna; H. Moraova, M. Kratka, & N. Stehlikova (Eds.), Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education, Vol. 5 (pp. 273-280). Prague, Czech Republic.

Uğurel, I. (2012). Non-thesis Master´s Level Pre-Service Mathematics Teachers' Conceptions of Proof, *Bolema*, *26*(42B), 715-742.

Ulep, S. A. (2007). Developing Mathematical Communication in Philippine Classroom. *Proceedings of APEC-TSUKUBA International Conference III, Innovation of Classroom Teaching and Learning through Lesson Study, Focusing on Mathematical Communication*. Tokyo and Kanazawa, Japan.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathematiques, 6*, 69-118.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 163–201). Hillsdale: Lawrence Erlbaum.

Vygotsky, L. S. (1962). *Thought and language*. Cambridge: MIT Press.

Wakefield, D. V. (2000). Math as a second language. *The Educational Forum, 64*(3), 272-79.

Wang, S. (2007). Research Process, Changes and Implementation of Mathematics Curriculum Standard of China. *Proceedings of APEC-TSUKUBA International Conference III, Innovation of Classroom Teaching and Learning through Lesson Study, "Focusing on Mathematical Communication*. Tokyo and Kanazawa, Japan.

Wood, L. A & Kroger, R. O. (2000). *Doing discourse analysis. Methods for studying action in talk and text*. Thousand Oaks, CA: Sage

Zeybek, D. (2003). Söylem ve Toplum, (Yayına Haz. A. Kocaman) Söylem Üzerine, (pp. 27-47). Ankara: ODTÜ Geliştirme Vakfı Yayıncılık ve İletişim A.Ş. Yayınları.

Author Information

Isikhan Ugurel
Dokuz Eylul University
Ugur Mumcu St. 135 Av. No.5, 35150 Buca-Izmir, Turkey
Contact e-mail: *isikhan.ugurel@deu.edu.tr*

Burcak Boz-Yaman
Mugla Sıtkı Kocman University
T-Block 48000 Mugla, Turkey

Appendix 1. Example questions from Prompted Discourse (PD)

1-   What is proof?
2-   Is proof important in mathematics?
3-   Why do we prove something in mathematics?
4-   What are the methods for proving?
5-   Who can prove?
6-   A theorem could have how many proofs?
7-   Is there any benefit in doing proof for you (student)? If yes, what are they?
8-   How important is doing proof to you?
9-   What is theorem?
10-   What is axiom?