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When the Fractional Cookie Begins to Crumble: Conceptual Understanding of Fractions in the Fifth Grade

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Abstract

When teachers conduct a universal screening in mathematics, they identify students who are struggling with mathematical content and adjust their instruction. In this mixed-methods study in Kolkata, India, teachers piloted a screening tool at the beginning of the academic year in 5th grade to determine students' (n = 171) understanding of fractional number sense. The results of the screening show that the majority (70%) of students in fifth grade are in need of more instruction in foundational fraction concepts. Common misconceptions on screener items are shared, as well as strategies for improving fraction instruction to prevent future errors.

Introduction

Students may feel overwhelmed or exceedingly bored in math classes – completing numerous fraction exercises in class or for homework, without really understanding the concepts. Students may begin to believe that they only have to be obedient and memorize the steps and methods the teacher tells them, without ever truly understanding fractions (Boaler, 2015).

Knowledge of fractions and other types of rational numbers is paramount for learning future concepts of algebra and geometry (National Mathematics Advisory Panel, 2008). In fact, Siegler et al. (2012) found that knowledge of fractions and whole-number division in primary school predicts math achievement in secondary school more than whole-number addition, subtraction, and multiplication; verbal and nonverbal IQ; working memory; family education; and family income. Misconceptions with fractions stem from lack of conceptual understanding (Fazio & Siegler, 2011).

Universal screening determines students' current conceptual understanding and can be used by teachers to decide instructional planning based on students' needs (Witzel & Little, 2016). Researchers have been developing early childhood screening tools to predict math difficulties in the early grades (Krasa & Shunkwiler, 2009). In fact, all students can be screened as early as kindergarten for difficulties in mathematics, including some tasks which are powerful predictors of math learning disabilities (MLD) (Desoete et al., 2009; Griffin & Case, 1997). However, no such tools are being used in elementary schools in India for mathematics, let alone for fractions.

This study investigates the use of a screening tool to examine primary students' mathematical thinking about fractions in Kolkata, India. The screening tool for students in 5th grade (referred to as 5th standard in India) was piloted in June and July 2015. The screening assessment tests grade-level skills and is brief in length. One of the goals of the screener is to identify students who are *on target, in need of some support,* and *in need of intensive support* in mathematics so teachers can better detect students who need remediation and intervention in mathematics at an early stage in the learning process (Winterman & Rosas, 2014).

A screening tool empowers teachers when they are trained in interpreting common misconceptions in mathematical understanding and identifying students who need help or intervention earlier than waiting for a diagnosis of learning disability (Karande, Sholarpurwala, & Kulkarni, 2011). A screener will reveal qualitative differences between students and their math abilities, which may help teachers understand the heterogeneity of students' math abilities.

The study focuses on the following research questions:

• What information can be gathered by teachers to ascertain Indian students' grade-level math skills and fractional number sense using a screening tool?
• What error patterns emerge with fractions among 5th standard students?
• How can teachers use a screening tool to change their instruction and intervention techniques to support students' understanding of fractional concepts?

Instruction with Fractions

Mathematics is like "a house of cards," since each concept requires "the coordination of lower level interrelated skills, each of which is itself grounded on very basic conceptual and procedural knowledge" (Rousselle & Noël, 2008, p. 498). Achievement in mathematics is dependent on strong number sense. Understanding numbers and their magnitude, quantity, and relationships to other numbers forms the foundation for later computation and application of fractions (Witzel & Little, 2016). Success with fractions and fractional computation is closely related to achievement in Algebra I (Van de Walle, Karp, & Bay-Williams, 2016). The acquisition of knowledge about fractions is a crucial process in numerical development (Siegler, Thompson, & Schneider, 2011). Specifically, recent studies have shown that the knowledge of fraction magnitude is central in continued math development (Resnick, Jordan, Hansen, Rajan, Rodrigues, Siegler, & Fuchs, 2016; Siegler, Thompson, & Schneider, 2011). Overall, number sense is foundational for fractions and higher-level math, including algebra.

Experiences with fractions should begin no later than first grade (Van de Walle, Karp, & Bay-Williams, 2016). Research has shown that students should be instructed in fractions using measurement activities, number lines, manipulatives, and visual representations to foster deep conceptual understanding (Siegler, et al., 2010; Fazio & Siegler, 2011). Additionally, teachers can use a variety of representations, such as area models, length/linear models, and set models, to deepen students' understanding of fractions (Van de Walle, Karp, & Bay-Williams, 2016).

Students must develop understanding of the magnitude of fractions, in addition to knowledge that fractions are equal parts of a whole. Number lines illustrate that fractions are numbers with magnitude and can be compared to other numbers, such as whole numbers (Fazio & Siegler, 2011; Siegler, Thompson, & Schneider, 2011). As students develop conceptual understanding of fractions, they will understand why the computational procedures work and make sense (Fazio & Siegler, 2011).

In the range between 0 and 1, students learn the magnitudes of benchmark fractions ($\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$) to assist with understanding fraction magnitude more generally (Siegler, Thompson, & Schneider, 2011). Having a sense of what the answer might be close to, based on fraction magnitude, will allow students to reject unreasonable answers. For example children might reject the procedure that produces arithmetic errors of the form $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ if they recognized that adding a quantity to $\frac{1}{2}$ cannot produce a quantity that is smaller than both addends. This could lead them to try other procedures and test whether their answer made sense (Siegler, Thompson, & Schneider, 2011).

Students in the United States have typically been introduced to fractions through area models, such as cookies, pizza, and brownies. (Freeman & Jorgensen, 2015). With the Common Core State Standards in the United States, students are now conceptualizing fractions as a quantity on a number line. Linear representations of fractions, such as number lines, are emphasized in elementary classrooms in high-achieving countries, such as Japan, China, and Korea (Lewis & Perry, 2017). Students can be taught to interpret $\frac{1}{5}$ as one of five slices of pizza (part-whole interpretation), but also think of $\frac{1}{5}$ as one-fifth of the distance from zero to one on a number line, as is done in the high-achieving countries mentioned above (Siegler, Thompson, & Schneider, 2011).

In the mathematics classroom, teachers are the most important resource for students and have the most impact on their mathematical learning (Boaler, 2016; Darling-Hammond, 2000). Teachers can use multiple representations to build conceptual understanding of fractions that are embedded in multiple real-life contexts. Teachers can also facilitate mathematical discussion by having students to explain their numerical reasoning and justify their strategies and thinking to their peers, (Humphreys & Parker, 2015).

Common Misconceptions with Fractions

Learning about fractions presents a tremendous shift for students' thinking. Students have difficulty moving from whole numbers to fractions, in part, due to the lack of focus on fractions as "numerical entities" (Siegler, Thompson, & Schneider, 2011, p. 274; Wynn, 1995, p. 176). Although fraction instruction begins in early elementary school, even high school and community college students often confuse properties of fractions and whole numbers (Siegler, Thompson, & Schneider, 2011; Schneider & Siegler, 2010; Vosniadou, 2014).

Students around the world struggle with fractions, and for some students, these difficulties persist through 8th grade (Mazzocco et al., 2013; Fazio & Siegler, 2011; Hecht & Vagi, 2010). Students often misinterpret fractions, due to the fraction bar, or vinculum, that appears between the numerator and denominator. Students often fail to see fractions as a quantity, which can be represented on a number line (Witzel & Little, 2016).

When students do not develop strong conceptual understanding of fractions, they may memorize the fraction algorithms to obtain correct answers. However, these students do not have an understanding of fraction magnitude and how the fractions are being manipulated, so their rote memorization becomes inaccurate in the short- and long-term. When students memorize procedures, they have a difficult time applying their knowledge to novel situations. In math textbooks, students are typically shown perfect examples and asked to practice isolated calculations or exercises over and over. Students may learn a method or procedure, but are unable to apply the math to a meaningful situation (Boaler, 2016). When students fail to understand the concepts behind fractions, or other math topics, misconceptions form. When students have difficulty understanding the conceptual underpinnings of fractions, teachers may get frustrated and focus on the procedures and memorized tricks to obtain the correct answer, while students don't understand why and if their answer makes sense (Schwartz, 2016). When students estimate and know whether or not their answer makes sense, they are more accurate in remembering the procedures over time (Siegler, Thompson, & Schneider, 2011).

Fraction Instruction in India

In India, schools are associated with various boards, or curricula, such as the Indian Certificate of Secondary Education (ICSE) Board and the Central Board of Secondary Education (CBSE). Education is a responsibility of both the national and state governments. The National Council for Educational Research and Training (NCERT) is an advisory body, and states choose to adopt or adapt the recommendations of NCERT, since the context varies considerably from state to state (M. Jain & K. Sharma, personal communication, July 5, 2013).

Table 1.Textbook pacing and NCERT syllabus for 4^{th} standard

ICSE curriculum topics (Chaudhuri, 2001)	Syllabus for math, Class 4 (NCERT, 2006a)
Proper and improper fractions	Identifies half, one-fourth and three-fourths of a whole.
Mixed fractions	Identifies the symbols $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}$
Conversion of improper fractions into mixed fractions	Explains the meaning of $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}$
Conversion of mixed fractions into improper fractions	Appreciates equivalence of $\frac{2}{4}$ & $\frac{1}{2}$; $\frac{2}{2}, \frac{3}{3}, \frac{4}{4}$ and 1 whole
Equivalent fractions Test for equivalent fractions Simplest or lowest form of a fraction Like and unlike fractions Addition of fractions (like and unlike, mixed) Comparison of fractions Subtraction of fractions Word problems Fractional parts of different quantities	

NCERT has developed a syllabus for mathematics for grades 1-5, which should inform textbook creators. Textbooks play a critical and prominent role in Indian math classrooms, as well as classrooms around the world (Reys & Reys, 2006; Doabler, Fien, Nelson-Walker, & Baker, 2012). When compared with the NCERT syllabus, the textbooks used by ICSE private schools in Kolkata are teaching advanced fraction concepts, ahead of when NCERT advises the concepts be introduced. According to the publisher, the Modern School Mathematics textbooks for Classes 1 - 5 cover the revised syllabus of NCERT and cater to schools affiliated with any board (Orient Blackswan, n.d.). However, the NCERT syllabus for standard 4 and 5 are completely different in regards to fraction instruction. Table 1, above, outlines the differences in 4th standard instruction on fractions while Table 2 contacts the 5th standard expectations. By fifth standard, Indian students are tested on fraction magnitude, fraction operations, and fraction relationships (mixed numbers and percentages) (Rao, Pearson, Cheng, & Taplin, 2013). The topics that are covered in the 4th standard ICSE textbook (Chaudhuri, 2001), such as addition and subtraction of fractions, are covered in the NCERT syllabus two years later, in 6th standard (NCERT, 2006b).

Table 2.Textbook pacing and NCERT syllabus for 5th standard

ICSE curriculum (Gopal, 2002)	Syllabus for math, Class 5 (NCERT, 2006a)
Multiplication of fractional numbers	Finds the fractional part of a collection.
Multiplicative inverse of a fraction	Compares fractions
Division of fractional numbers	Identifies equivalent fractions
Properties of multiplication and division of fractional numbers	Estimates the degree of closeness of a fraction to known fractions ($\frac{1}{2}, \frac{1}{4}, \frac{3}{4}$, etc.)
Decimals: all operations	Uses decimal fractions in the context of units of length and money
	Expresses a given fraction in decimal notation and vice versa

Again, topics that are covered in the 5th standard ICSE textbook (Gopal, 2002), such as multiplication and division of fractions, are covered 2 years later in the NCERT syllabus (NCERT, 2006b). Typically, students from private schools score between 10 and 25 percent higher on standardized tests, as compared to public school students in India (Rao, Pearson, Cheng, & Taplin, 2013). Exposure to more advanced concepts may be one factor in this performance gap.

Framework

The overarching theoretical framework for the study is rooted is constructivism and the importance of misconceptions (Piaget, 1970; Olivier, 1989). According to constructivism, a student learns because of an interaction between existing ideas and new ideas, as well as experiences. Students organize and structure knowledge based on units of interrelated ideas and concepts, called schemas. As students attempt to integrate new knowledge with their existing schemas, they may overgeneralize their previous knowledge, such as whole number reasoning, to a new domain of knowledge, like fractions. When students incorporate their new knowledge, they may apply it to previous knowledge in a way that makes sense to them, but is mathematically incorrect. When teachers interpret students' errors as their rational and meaningful way to cope with new mathematical ideas and understanding, rather than the student making a silly or stupid mistake, they can use the errors as an opportunity to learn (Olivier, 1989). Teachers can create a classroom environment that normalizes errors as part of the learning process and engage the students in mathematical discourse about the errors and misconceptions in order to ensure students have deep conceptual understanding, while correctly connecting new knowledge to their previous knowledge (Olivier, 1989).

Misconceptions will never be entirely avoided, but teachers can intervene before the misconception becomes deeply rooted. First, teachers must understand why their students are making errors or how they have developed misconceptions before they can address them and develop interventions to promote true understanding (Olivier, 1989; Harbour, Karp, & Lingo, 2016). By using formative assessment tools, such as a universal screener, teachers can begin to discover the root of their students' misconceptions and errors. Teachers must first gather evidence on a mathematical concept or skill, like fractional number sense, and then use the information to shape and guide their instruction (Harbour, Karp, & Lingo, 2016). A fraction sense screener is one way that teachers can collect evidence about students' thinking about fractions, adjust their instruction, and help students develop deep conceptual understanding with fractions before moving onto more advanced fractional computation.

Method

Setting and Participants

The study took place in Kolkata, the third largest urban area in India (following Mumbai and Delhi). Kolkata (formerly Calcutta) has a population of more than 14 million and is located in the state of West Bengal (Indian Population Census, 2011). After receiving Institutional Review Board approval, Breaking Through Dyslexia (BTD), a non-profit educational organization in Kolkata, recruited 171 fifth standard students in private primary schools. Any students currently in the BTD network, currently receiving remedial services in the 5th standard, were also asked to participate. Classroom teachers were asked to rate each student's math ability as are on target, in need of some support, or in need of intensive support in mathematics. Although information was handed out to equal numbers of males and females in each school, more females participated in the study (males: n = 82, females: n = 89), with the exception of St. Mary's School (psudeonym), an all-girls school. This sample is above 40 students, which should show a normal distribution of scores. The average age of the participants was 10 years, 0 months.

All schools in the sample are affiliated with the ICSE board, except for Lotus School which follows the CBSE board. These schools were chosen and asked to participate because they had dyslexia awareness programs conducted by BTD in their school over the past few years. One school, Adarsh School, is an "integrated school," in which students with special needs (20-25% of the total class) learn alongside typically achieving students (school website). All of the schools, as well as BTD, are located in South Kolkata. A summary of the participants is located in Table 3. The students denoted as "other" were five students that participated from the BTD network that attend other English medium private schools in Kolkata.

Table 3. Participating private schools in Kolkata

School name (pseudonym)	Language of instruction	Board affiliation	Total number of students	Number of 5th std. males in the sample	Number of 5th std. females in the sample	Average age of participants	Monthly tuition fees in rupees (INR)	Average class size for fifth std.
Vidyamandir	English	ICSE	2,090	21	12	10.0	3,000	30
Balkrishna	English	ICSE	3,380	7	10	9.8	2,000	50
St.Mary's	English	ICSE	1,500	0	31	10.5	2,000	45
Adarsh	English	ICSE	445	17	14	10.0	2,500	25
Sunrise	English	ICSE	1,465	16	7	10.1	5,000	30
Lotus	English	CBSE	1,878	18	13	9.8	3,500	35
Other	English	CBSE/ISCE	N/A	3	2	10.5	N/A	N/A

In Kolkata, the academic year runs from June to April. Students take exams to complete the academic year in the month of last week of February and early March. Then, teachers begin to teach the material in the next standard during the first week April (after a break of 10 days). Summer vacation (40 days) occurs between mid-May and mid-June, when schools re-open. A test/assessment is given on opening day to ensure all students return after the holidays. Therefore, students in this sample, would have been exposed to fifth standard materials since the beginning of April, but with a break of 40 days, when the screener was administered between mid-June to the end of July 2015. It was important to be aware of the history effect, or recognizing that having the students take the screener at different times could impact the results – the students who take the screener last will have learned more math and been back in the academic environment longer. Therefore, the students were all given the screener within a one-month period. The sample is only made up of students from private schools that use English as the language of instruction; no students from government or vernacular-medium (Bengali, Hindi, etc.) schools participated. All schools were in the urban environment of metropolitan Kolkata.

Procedures

A mixed methods approach was used to gain in-depth knowledge of student mathematical thinking. Quantitative data about students' correct or incorrect answers was collected through various assessments, but qualitative data was also collected to determine the nature of their responses and their strategies used to find the answers. Both procedural fluency and conceptual understanding of fractional number sense were assessed.

The study was intended as a one group post-test only design. We wanted to measure how much math students know as they start 5th standard. The study and screening tool were developed as a starting point, not to develop a standardized instrument. First and foremost, we wanted to begin to understand students' mathematical thinking and to help teachers realize the heterogeneity of students' math abilities. A possible outcome of this screening tool is for teachers to group or classify students into small groups for instruction at the beginning of the school year. Therefore, the focus was not on test-retest reliability (giving the students the same test two weeks later) or inter-rater reliability (two examiners scoring the test).

After securing parental consent and student assent, students in English-medium schools completed the Woodcock Johnson IV Test of Achievement Calculation and Math Fluency subtests as a standardized measure (and as a warm-up for the screener). All students then completed the exploratory math screener at the fifth standard level, which consisted of 10 questions. The screener was constructed by the researcher, based on the NCERT (2006a) Syllabus for Classes at the Elementary Level, as well as other sources (Lewis, 2014; Petit et al., 2010; Tobey & Minton, 2011; Hecht & Vagi, 2010; Tobey & Fagan, 2014; Teaching Channel, 2016). Overall, the screener measured skills focused on fractional number sense. Number lines were included on the screener because they can represent "conceptual underpinnings" of various components of number sense, including number comparison and number transformation, and also provide students with a schematic image (Krasa & Shunkwiler, 2009, p. 28).

All assessments were untimed, with the exception of the Math Fluency subtest. The focus was on untimed assessments, since findings on math performance are stronger on untimed items than on timed items (Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013). Paper and pencil were used for problem presentation and responses. Students were given an unlimited amount of time to complete the screener and took an average of 19 minutes. The range of time students took was between 10 minutes and 31.5 minutes. Depending on the space available at the school, students completed the assessments in small groups of 6-10 students at a time, during school hours. Results of students' assessments were compared with the teachers' rating of performance (high, average, low-performing). Students' performance on the screeners were analyzed for common misconceptions in order to create a guide to help regular education teachers interpret students' errors and adjust their teaching or consider different teaching strategies. Parents of participating students completed a survey (available in English) for descriptive statistics of the sample. Since this study was conducted in private schools, the policy documents of the educational boards were analyzed, since private schools are independent of state and national policy because they do not accept any government funding.

Results and Discussion

Teachers can use the screening tool as a first step in collecting baseline information regarding their students' math abilities, as well as begin to understand the variability of fractional number sense in their students. We began this research with the notion that the screener would help teachers identify students who are on target, in need of some support, and in need of intensive support in fractions. Contrary to expectations, the screener revealed that the majority of 5th standard students in the sample may be in need of intensive support in understanding fractions and fractional computation. Although the majority of Indian students in this sample have average to above-average skills as compared to the U.S. norms as revealed by the Woodcock Johnson IV Test of Achievement calculation and math fluency subtests, their performance on the fractional number sense tasks was surprising and quite varied. The range of screener scores is shown by school in Table 4. There is not a normal distribution since the results are highly positively skewed, indicating a greater number of smaller values. To account for the skewed distribution, a square root transformation was performed on the total screener score.

Table 4. Range of screener scores (total of 14)

School name (pseudonym)	High score	Low score
Vidyamandir	8.5	0
Balkrishna	12	2
St.Mary's	7	0
Adarsh	12	0
Sunrise	5	0
Lotus	7	0
All participants combined	12	0

At each school, teachers were asked to rate each student in their class as high-achieving, average, or needing support. Results, by school, are shown in Table 5.

Table 5. Teacher rating

School name (pseudonym)	High-achieving	Average	Needs support
Vidyamandir	5	14	13
Balkrishna	8	10	0
St.Mary's	15	12	3
Adarsh	9	14	8
Sunrise	12	9	2
Lotus	10	16	4
All participants combined	59	75	30

Because some of the students were not tested in schools, there were 7 students from which we did not collect teacher ratings. For this reason, the number of participants used was n = 164. An analysis of variance (ANOVA) was used to test differences among groups (teacher rating). An ANOVA is robust against the normality assumption. A protected LSD was performed to determine pairwise comparisons. There are significant differences in screener score means between students who received a teacher rating of high achieving (1), average (2), and needing support (3). To control for inherent difference among schools, adjustments were made for school difference. A one-way ANOVA revealed significant differences among the groups (teacher rating) across the sample ($p = <.0001$). The results are presented in Table 6 and Figure 1. Error bars represent the standard error of the mean.

Table 6. Screener scores according to teacher rating

Teacher Rating	Number of students	Screener Score Mean (Standard Error)
High achieving (1)	59	3.864 (0.326)
Average (2)	75	2.993 (0.255)
Needs support (3)	30	1.100 (0.216)

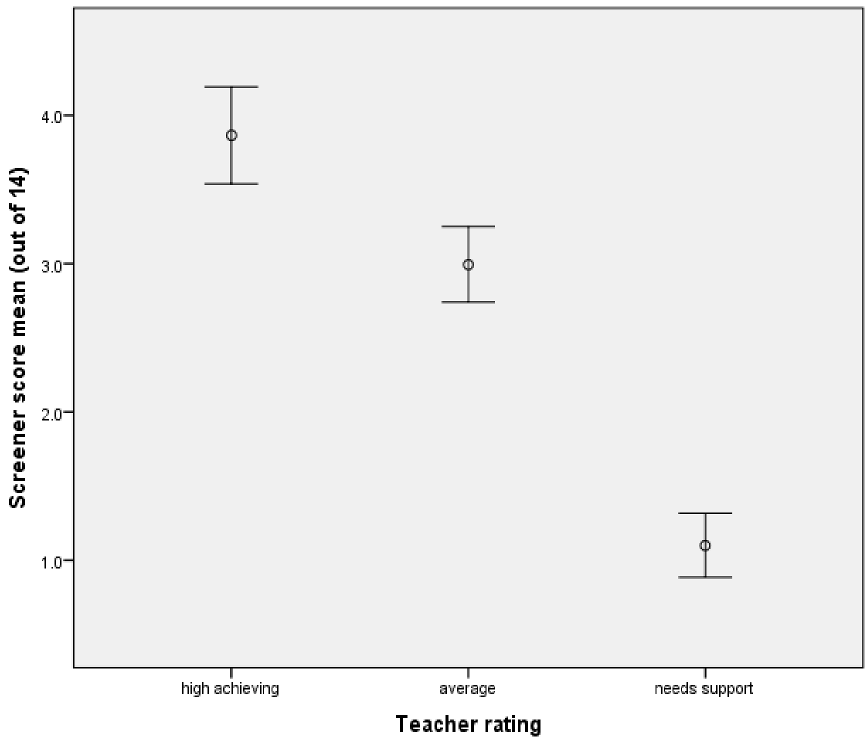

Figure 1. Differences in screener scores among groups

In order to analyze students' errors, students' strategies were coded, in addition to their correct or incorrect response. The most common misconceptions and errors for each question are listed in Table 7.

Table 7. Common misconceptions by screener question

Screener question	Most common misconception	Completed by number of students (out of 171)
1: Draw a picture of $\frac{1}{2}$	Student partitions shape into 2, but does not shade	n = 36 (21.1%)
2: Least to greatest: $\frac{1}{2}, \frac{1}{3}, \frac{4}{5}$	Uses whole number reasoning: $\frac{1}{2}, \frac{1}{3}, \frac{4}{5}$	n = 62 (36.3%)
3: The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to	20	n = 81 (47.4%)
4: Which is greater? $\frac{2}{5}$ or $\frac{3}{10}$	$\frac{3}{10}$ is greater (Cadbury)	n = 120 (70.2%)
5: Shade $\frac{3}{8}$ of the figure	Shaded 3 of the 16 boxes	n = 90 (52.6%)
6: How much plum cake is left over?	No attempt	n = 38 (22.2%)
7: : $\frac{1}{2}+\frac{1}{3}=$ ______	$\frac{2}{5}$	n = 68 (39.8%)
8: Mark and label $\frac{1}{3}$ and $\frac{1}{4}$ on the number line	$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$: whole number magnitude	n = 71 (41.5%)
9: Name the point shown on the number line ($\frac{1}{10}$)	No attempt	n = 41 (24%)
10: Mark and label $\frac{3}{4}$ on the number line (between 0 and 4)	Midpoint of 3 and 4	n = 47 (27.5%)

Fractional Number Sense and Error Patterns

Students in this sample exhibited many common misconceptions in regards to conceptual understanding of fractions in general, as mentioned in Table 8 (SciMathMN, 2016; Gojak & Miles, 2015).

Table 8. Common misconceptions with fractions

Fraction topic	Misconception
Equal parts of a whole	Students may think that any object divided into parts, regardless of size, is a fractional piece of the whole
Comparing fractions	Students may draw models which contain the correct number of pieces in the whole, while disregarding that they are not the same-sized pieces or same-sized wholes.
Comparing and ordering fractions	Students may apply whole number understanding – the fraction with the larger denominator is the larger fraction (one-fourth is larger than one-half, since 4 is larger than 2).

Several common error patterns emerged among 5th standard students. The most common misconception in drawing one-half was partitioning shape into 2, but not shading one of the two equal parts, as show in Figure 2.

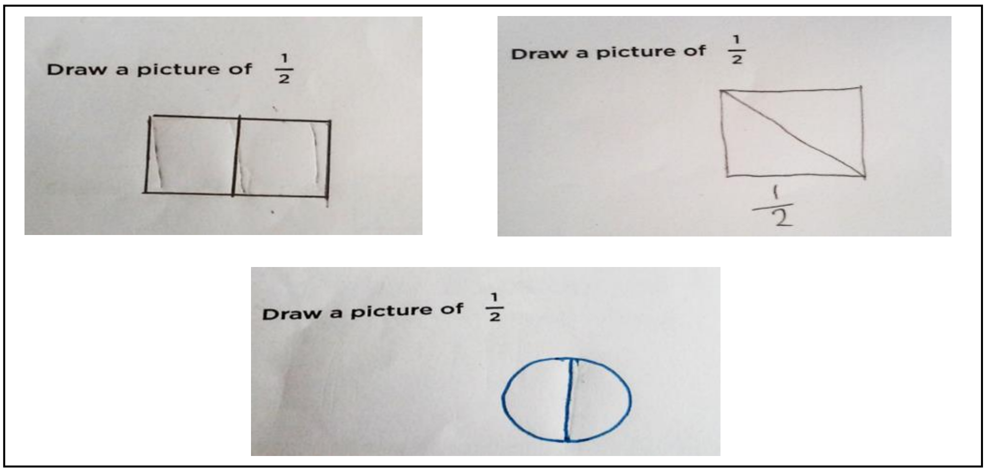

Figure 2. Partitioned the shape into two without shading

When comparing $\frac{2}{5}$ and $\frac{3}{10}$, many students were not aware that the relative size of the whole matters when comparing fractions with models. In Figure 3, students drew two different models, some of different sizes and shapes.

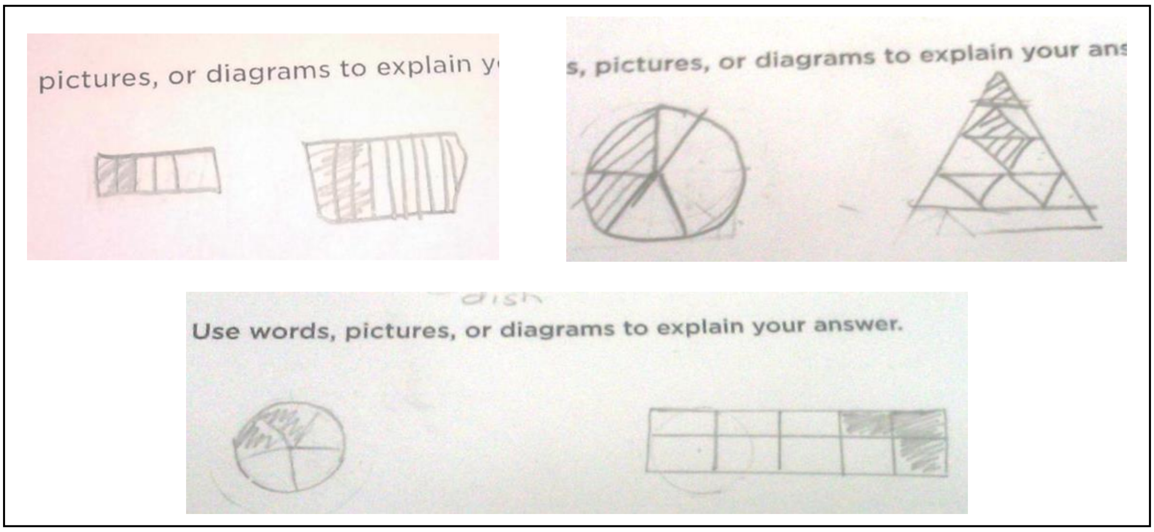

Figure 3. Incongruent wholes to compare the fractions $\frac{2}{5}$ and $\frac{3}{10}$

Forty-four students (25.7%) did not attempt to compare the fractions. The most common reason for an incorrect answer was that 62 students (36.3%) attempted to draw models, but they were incorrect. Some models showed that students may think that an object divided into parts, regardless of size, is a fractional piece of the whole. In Figure 4, a student was unable to draw equal-sized parts for fifths and tenths.

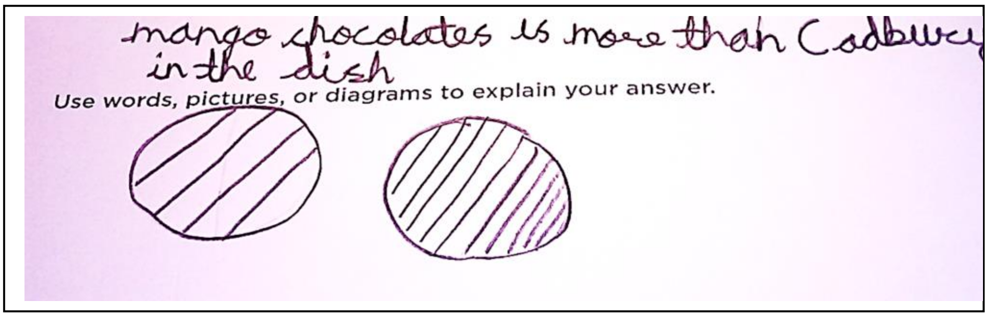

Figure 4. Similar wholes with unequal parts

While 51 (29.8%) students identified $\frac{2}{5}$ as more than $\frac{3}{10}$, only 8.2% (n = 14) of students were able to correctly explain why their answer was correct using words, pictures, or diagrams. Some students were extremely precise in their explanation, as shown in Figure 5.

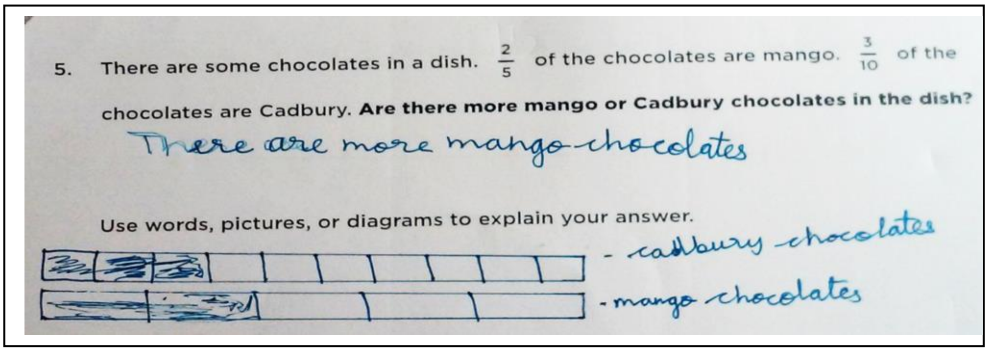

Figure 5. Same size wholes to compare $\frac{2}{5}$ and $\frac{3}{10}$

Students in this sample also exhibited difficulty with equivalent fractions. Only 29 students (17% of the sample correctly shaded 6 out of 16 boxes together, in rows and columns, not spread apart into two groups. Ninety students (52.6%) shaded only 3 boxes, shown in Figure 6.

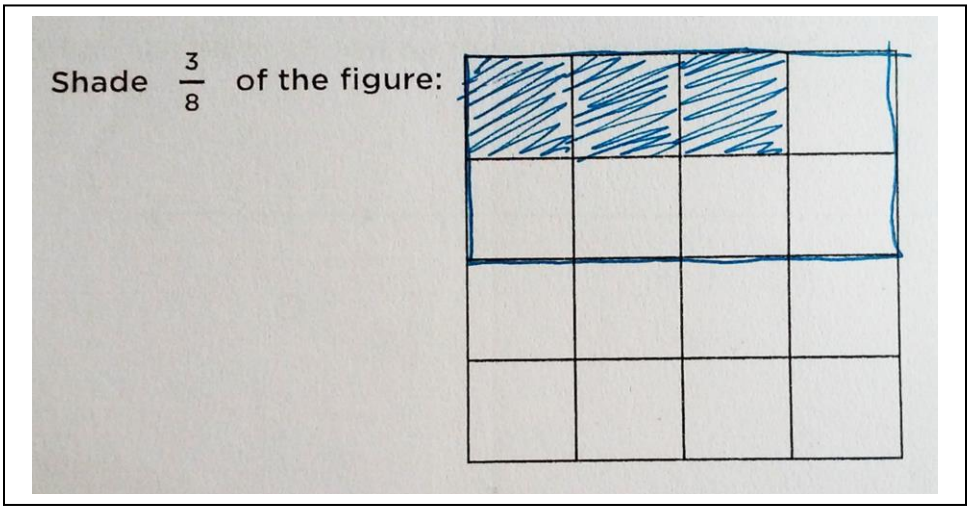

Figure 6. Ignoring the entire whole (sixteenths)

Students had difficulty comparing fractions. Both off and on the number line, students incorrectly applied whole number understanding to fractions, showing their thinking that the fraction with the larger denominator is the larger fraction (shown in Figure 7).

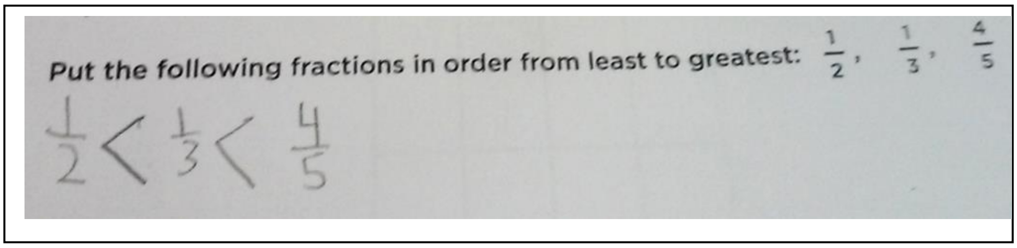

Figure 7. Using whole number reasoning to order fractions

Conversely, 8.2% of students (n = 14) used common denominators or found the lowest common denominator to determine where the numbers belong on the number line. Only four students (2.3%) used models (Figure 8) to correctly order the fractions. Most students looked at the examiners quizzically and said, “But, they are already in order from least to greatest.”

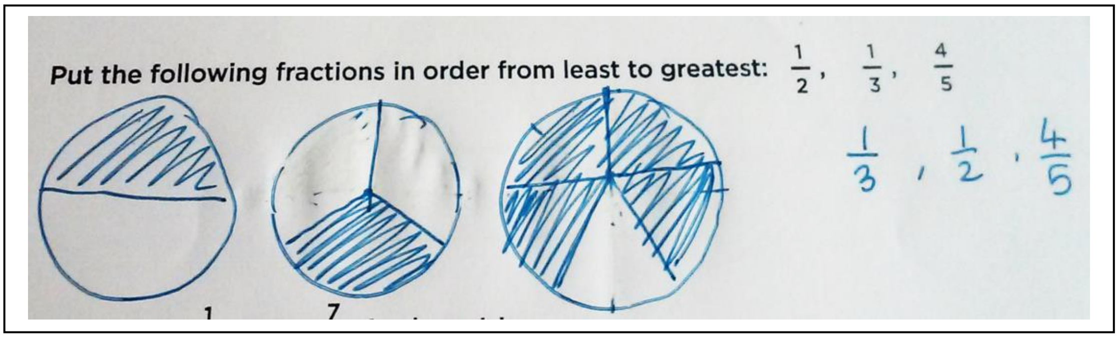

Figure 8. Using models to order fractions

On this problem, 57% (n = 97) of students correctly identified that $\frac{4}{5}$ was the largest fraction. However, only 22% (n = 38) of students correctly identified that $\frac{1}{3}$ was the smallest fractional amount listed. On a number line (Figure 9), students continued to apply whole number reasoning incorrectly when representing fractions.

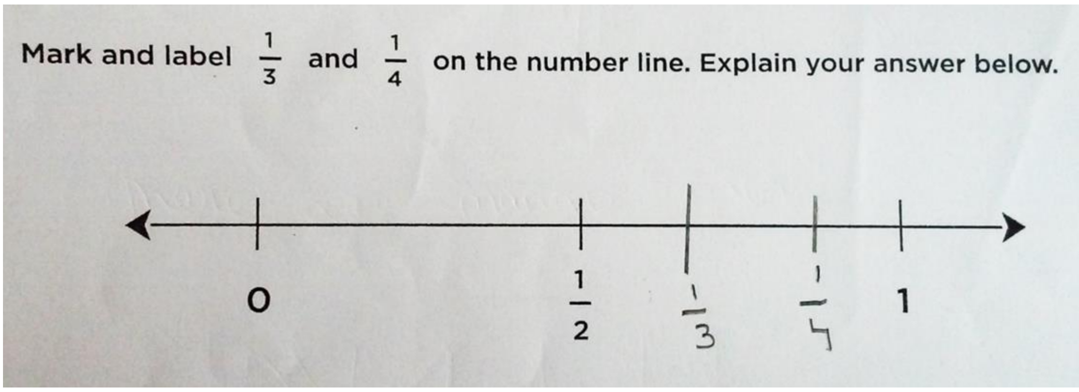

Figure 9. Whole number reasoning to sequentially order the denominators

To measure their ability to estimate fractions, students were asked to choose one of the following answers: The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to: a. 20; b. 8; c. $\frac{1}{2}$; d. 1**.** Of the students that did not identify the correct answer (d), 12.9% (n = 22) of students did not attempt the problem. The most common incorrect answer was a. 20, which 47.4% (n = 81) of the students chose. The next most common incorrect estimation was b. 8, which was chosen by 19.3% (n = 33) of the students. Students were unable to determine which answers would be unreasonable, given their lack of fraction magnitude knowledge.

Overall, students in this sample showed evidence of common misconceptions regarding fractions. There is a great deal of variability in their understanding of fractions and their magnitude. Even individual students showed inconsistency in their ability to compute and solve problems with fractions, as evidenced by Figure 10. This student was able to solve a word problem involving addition and subtraction of benchmark fractions by drawing a diagram, yet was unable to add two benchmark fractions in the next problem.

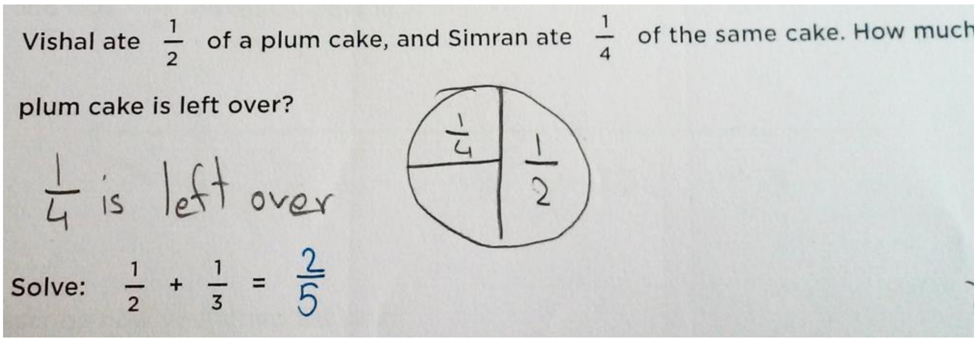

Figure 10. Inconsistent computation with fractions

The majority of students showed poor conceptual understanding of fractional number sense during this screening. Their errors reveal common misconceptions and gaps in conceptual and procedural knowledge. As their teachers analyze students' errors, they can consider ways to change their instruction to correct faulty understanding. Teachers can target the errors specifically, instead of broadly re-teaching the concepts. In the future, teachers adjust their instruction to anticipate the common misconceptions about fractions and to build stronger fractional number sense. Teachers can conduct a diagnostic interview to determine if the errors are related to misconceptions.

Textbooks

In a review of fourth standard textbooks used in Kolkata schools, there were few fractional models or representations. For instance, in an ICSE textbook, there were visual representations of fractions only for the first part of the chapter, which introduced the terms numerator and denominator (Chaudhuri, 2001). Once a new topic was introduced, e.g. equivalent fractions, there were no representations, only abstract procedures were presented. In the case of equivalent fractions, the only focus was on the rule, or procedure, to create equivalent fractions – multiply both the numerator and denominator by the same number.

In a fifth standard ICSE textbook by the same company, the chapter on fractions immediately jumps into multiplication and division of fractions after a short review of part-whole relationships, equivalent fractions (Gopal, 2002). All of the fraction multiplication representations are shown in circular area models. There is also no real-life application to introduce this concept. Students may not see fractions as meaningful when they are not learning them in context and through multiple representations. In schools in Kolkata, students move quickly to abstract mathematics. Even 4^{th} standard ICSE textbooks have few fractional models or representations (Chaudhuri, 2001).

Teacher Knowledge

Another factor that might have influenced the current findings is teachers' content knowledge (Moseley & Okamoto, 2008). According to Shulman (1986), teachers must have content knowledge about math, but they also need to have pedagogical content knowledge – knowledge of the best practices for teaching math to elementary students. Strong academic standards and quality curriculum are important to mathematics instruction, yet skilled teachers, who are confident in their mathematical abilities, are needed to engage and support students in order to learn mathematics (Ball, Hill, & Bass, 2005).

Following the screener of 5^{th} standard students at each school, we presented the results to teachers and administrators. Teachers became defensive at times, as if we were accusing them of students' errors. In some cases, the teachers critiqued the screener questions and the timing of the screener, insisting that they had not yet taught fractions by that point in the academic year (5^{th} standard teacher, personal communication, July 15, 2015). The screener is not necessarily assessing students on what they have been taught in school about fractions. Students should not need to study for the screener. The screener can be one tool to measure fractional number sense, and students will come to elementary classrooms with various degrees of number sense (and this cannot be controlled by the teacher). When presented with the results of their students, teachers mentioned that the syllabus is very heavy. It is difficult to teach all of the topics they are expected to cover (5^{th} standard teacher, personal communication, July 14, 2015).

Data on students' performance, perhaps collected through a screening tool, can be used as evidence to stimulate discussion and provide an opportunity for teachers to hone their craft of teaching. The screener may be perceived as a naming and blaming tool to be used against them and their teaching. The teachers' reactions in Kolkata are similar to teachers' reactions in a study of South African teachers (Shalem, Sapire, & Sorto, 2014). The screener should not be used as a teacher evaluation tool. In order for teachers to look at students' responses and learn from their errors, they need safe spaces to acknowledge their inadequacies. Applying the findings of error analysis can be difficult, if teachers feel threatened by the students' results, and if they are unsure how to address the error patterns. However, school climate and on-going professional development can normalize misconceptions and errors to a certain extent.

Through the constructivist lens, we view errors as students' attempt to construct their math knowledge. Misconceptions will never be entirely avoided. However, when teachers establish a classroom environment where errors are seen as an opportunity to learn and grow in our understanding of math, students may respond more positively to math and have less anxiety while engaging with mathematical content (Olivier, 1989). They can also begin to adopt a growth mindset and view mistakes as opportunities for your brain to grow (Dweck, 2006; Boaler, 2015). By using the screener, or other screening tools, a teacher uses students' errors to change his/her instruction, rather than attributing student performance solely to their teaching. Universal screening can be seen as assessment *for* learning more about students' current levels of understanding and helping teachers understand how they adjust their instruction, not an assessment *of* how much the students have learned (Boaler, 2015).

When we shared concrete teaching strategies, using manipulatives, teachers at some of the schools in this sample mentioned that they have similar items in Montessori classrooms in their school (5^{th} standard teacher, personal communication, July 14, 2015; Kindergarten teacher, personal communication, July 27, 2015). Teachers remarked that these materials were used in pre-school and kindergarten, but the use of concrete materials was not a hallmark of instruction in the primary classes. Students move to abstract mathematics, the class sizes are larger, the pace of instruction is faster, and the syllabus is longer.

Conclusion

The final analysis of this screener is different than initially anticipated. Instead of being a tool that grouped students by their level of understanding, the screener revealed many misconceptions that fifth standard students have with fractions. This research does not aim to blame teachers for students' misconceptions about fractions. Teachers in India are doing the best they can, with the training and resources they have been given. Teachers may feel incompetent when they are expected to implement new ways of teaching without proper training (Heifetz & Linsky, 2002; Vaillancourt, 2016). It is necessary to allow teachers to feel supported and equipped to use new methods for teaching fractions.

When students struggle in math, they do not need more practice with facts or methods. Students that struggle with fractions will not improve if will give them more problems to practice. Instead, they need targeted instruction that highlights the conceptual understanding of what fractions are, what they look like in various representations, how they relate to other fractional quantities, and why the procedures in the algorithms for adding and subtracting fractions work (Boaler, 2015).

This research study has limitations, considering it consisted of a screening tool which was administered only once. Therefore, it is similar to a one-time snapshot of the students' performance. Because students receive additional instruction in fractions during 5th and 6th standard, it would be beneficial to compare the students' results over time. A longer exposure to fractions may reveal greater understanding of fraction magnitude, as suggested by Resnick, Jordan, Hansen, Rajan, Rodrigues, Siegler, & Fuchs (2016).

This study was conducted with a relatively small sample size (n = 171) and the population was made up of students from private schools; no students from government or vernacular-medium (Bengali, Hindi, etc.) schools participated. The sample was taken from a middle-class and upper-middle class section of urban Kolkata. More research can be done to determine students' fractional number sense in marginalized populations in urban areas of India, as well as in rural areas. Also, this was an exploratory study and the screening tool is not normed.

The screener is not yet ready for use as a diagnostic instrument because it is not normed. Nevertheless, to enable further tests of its utility, the screener, along with a scoring rubric, and a list of potential misconceptions for each item can be provided upon request. When using the research-constructed screener, it is recommended to include a warm-up question when not giving the WJ-4 subtests. The warm-up question could be a precursor which leads to fractional magnitude on the number line. The author is interested in further feedback from this pilot in order to make adjustments to the screener.

Recommendations

Based on the findings of this pilot study, the following recommendations were shared with the teachers of the participants:

Table 8. Ways to develop fractional number sense

Teachers will…	Students will…
Conduct a universal screening for fractional number sense	Show their conceptual understanding behind the fractional procedures in words and representations
Supplement textbook material with concrete manipulatives and representational drawings	Explain their thinking in multiple ways and understand the reasoning behind the procedures
Facilitate mathematical discussion using probing questions	Make sense of problems and justify their reasoning
View errors as opportunities for learning	

NCERT could consider revising their math syllabus to include fraction experiences as early as first standard, since students need significant time and multiple experiences to develop conceptual understanding in this topic. Also, textbooks can incorporate a variety of fraction models in their fraction representations, including linear (number lines and fraction strips) and set models, in addition to the circular area model.

Further professional development for current teachers is necessary. Teachers need courses and workshops in fractional sense and representational models, and they also need demonstrations in their classrooms. When teachers see other math educators interacting with their students, it can be a powerful learning experience (Kenschaft, 1997). Teachers facilitate conversation about the ways the models are related and how they relate to real-life scenarios.

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