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Abstract

The main purpose of this study is to investigate mathematics teachers' mathematical thinking process while they are constructing the length of $\sqrt[3]{2}$ by paper folding. To carry out this aim, two teachers -who are PhD. students- were interviewed one by one. During the construction, it was possible to observe the consolidation process of Pythagorean and Thales Theorem. All interviews were videotaped and analyzed through descriptive methods, according to abstraction in context, characteristics of consolidation and mathematical habits of mind. It was found that both two teachers constructed the knowledge of Pythagorean and Thales Theorems before the study and also these knowledges were consolidated. In addition, it was determined that these two approaches (abstraction in context and mathematical habits of mind) were close and corroborated each other. Moreover, consolidation process corroborated mathematical habits of mind.

Introduction

It is important for both researchers and teachers to understand how students abstract knowledge deeply (Dreyfus, 2012). According to Skemp (1986), abstraction is an activity depending on experiences, and it is an end-product. In addition, Davydov (1990) stated that theoretical concepts are produced by a mental activity. From the point of view, it can be said that abstraction is a mental activity and a mental activity occurs out of a thinking process. Thus, it is important to understand how a person thinks. Driscoll, DiMatteo, Nikula and Egan (2007) stated that a person needs to gain some kind of habits of mind to improve mathematical thinking. Thus, if a person's mathematical habits of mind (MHM) are revealed, his/her abstraction can be explained clearly. It is noticed that abstraction in context and mathematical habits of mind (both geometrical and algebraic) which stated by Driscoll et al. (2007) can be used together to understand someone's mathematical thinking.

It is possible to construct $\sqrt{2}$ by ruler and compass while it is impossible for $\sqrt[3]{2}$. Doubling the cube ($\sqrt[3]{2}$) which is also known as "Delian Problem" is an ancient geometric problem. To overcome this problem, paper folding can be used and this length can be constructed by a technic of paper folding called "Peter Messer's Solution" (Krier, 2007). In order to construct the mentioned length some theorems and rules should be known and consolidated as Pythagorean and Euclidean Theorem and similarity. Consolidation is an epistemic action which has a place in abstraction in context; RBC+C model (Dreyfus & Tsamir, 2004). In addition, Wiles (2013) stated that levels of MHM can be observable by paper folding and by this way thinking way of the person can be explained. When looked from this viewpoint, it seems a requirement to investigate the length $\sqrt[3]{2}$ to reveal a person's mathematical thinking in terms of RBC+C and MHM. In the light of these two theories, the main purpose of this study is to investigate mathematics teachers' mathematical thinking process while they are constructing the length of $\sqrt[3]{2}$ of which solution is impossible by using algebra in the history of mathematics but it is possible by paper folding. Indeed, our focus is not only on the constructing the mentioned length. For us, it is more important to explain how participants go through the process in terms of mathematical thinking while constructing it. In this process, they should know Pythagorean and Thales Theorems to construct the length. So, there are three sub-goals:

1) To analyze the construction process of $\sqrt[3]{2}$ in terms of RBC+C model which has epistemic action in accordance with abstraction in context and MHM.
2) To analyze the consolidation process of Pythagorean Theorem and Thales Theorem.
3) To determine whether these two approach (RBC+C model and MHM) corroborate each other or not during the abstraction process.

Studying with teachers is important; because teachers design learning activities according to his/her habits and experiences; and this situation affects students' learning and abstraction process. Misfeldt and Johansen (2015) carried out a research on mathematicians' practices in selecting mathematical problems. And they claimed that education and their teaching method were affected by their problem selecting process. As they are, we think that not only the selecting process but also the problem solving process of mathematicians and mathematics teachers is an important point to improve mathematics education and we should reveal mathematics teacher's construction process and their MHM. Because of this reason, in this study, we preferred to study with teachers.

Theoretical Framework and Literature

In this study we would like to reveal whether there is a link between RBC+C and MHM or not. We preferred to carry out this study with the help of paper folding activities. All of these are explained as follows.

Abstraction in Context (RBC+C Model)

Considering from the constructivist point of view, emergence of new mathematical constructs depends on the fact that previous constructs should be understood exactly and the relationship between constructs should be linked well (Dreyfus, 2012). To observe this construction process, Hershkowitz, Schwarz and Dreyfus (2001) submitted abstraction in context which is a model that included the observable stages; Recognizing (R), Building-with (B) and Constructing (C). The model is called RBC. The authors mentioned the importance of consolidation of the newly emerged structures and by adding the consolidation process, the model was called RBC+C model.

A construct can be consolidated when the knowledge is recognized and used in the further activities and consolidation is a never-ending process (Dreyfus, 2012; Anabousy & Tabach; 2015). Students become aware of construct and they decide to use it immediately (Anabousy & Tabach; 2015). Dreyfus and Tsamir (2004) showed that consolidation can be identified in terms of psychological and cognitive characteristics of self-evidence, confidence, immediacy, flexibility and awareness. Tsamir and Dreyfus (2005) explained the characteristics of consolidation as follows:

> Immediacy refers to the speed and directness with which a structure is recognized or made use of in order to achieve a goal; self-evidence refers to the obviousness that the use of a structure has for the student; obviousness implies that the student feels no need to justify or explain the use of the structure, though (s)he is able to justify and explain it. Self-evidence is directly related to the confidence or certainty with which a structure is used. Confidence refers to be sure about activity and not to be in doubt. Frequent use of a structure is likely to support the establishment of connections, and thus contribute to the flexibility of its use. A student may be quite proficient in using a structure, even using it flexibly, but without being consciously aware that s(he)is doing so. The awareness of a structure enables the student to reflect on related mathematical and instructional issues, add to the depth of her or his theoretical knowledge and power and ease when using the structure.

Most of the researchers carried out researches on abstraction in context 'Recognizing', 'Building-with' and 'Constructing' as epistemic actions, (Altun & Kayapınar, 2011; Hershkowitz, Schwarz, & Dreyfus, 2001; Dreyfus, Hershkowitz, & Schwarz, 2001; Kidron & Dreyfus, 2010), while several researchers emphasized the consolidation process (Dreyfus & Tsamir, 2004; Tabach, Hershkowitz & Schwarz, 2006; Monaghan & Ozmantar, 2006; Dreyfus, Hadas, Hershkowitz & Schwarz, 2006) and characteristics of consolidation (Dreyfus & Tsamir, 2004, Tsamir & Dreyfus, 2005; Anabousy & Tabach, 2015).

Dreyfus and Tsamir (2004) carried out a study which revealed the characteristics of consolidation that emerges from a sequence of interviews about the comparison of infinite sets with a talented student. They stated the mentioned characteristics as self-evidence, confidence, immediacy, flexibility and awareness which are the psychological and cognitive components. They proposed to take the combination of five characteristics as definition for consolidation process.

Dreyfus et al. (2006) tried to identify mechanisms to consolidate the recent knowledge constructs and they analyzed the processes of abstraction of a group of students working together in a classroom on tasks from a unit on probability studied. They stated that constructing and consolidating processes are often nested processes. Monhagan and Ozmantar (2006) worked with a seventeen-year old student with tasks related to the absolute value functions. They discussed that an abstracted knowledge is a consolidated construction that can be used to create new constructions and they gave evidence that an unconsolidated construction cannot be used to create new theoretical knowledge. This statement is a result of Dreyfus (2012)'s statement that a construct can be consolidated when the knowledge is recognized or used in the further activities.

A study of which method was Documenting Collective Activity (DCA) approach, as commonly used to establish normative ways of reasoning in classrooms, was carried out by Herkowitz, Tabach, Rasmussen and Dreyfus (2014). They emphasized DCA and RBC+C are different methodologies but closely related to the classroom learning process. They tried to identify and understand the process of governing shifts of knowledge. They used DCA to analyze whole-class discussion while they used RBC+C for analyzing groups' work. Anabousy and Tabach (2015) carried out a study with two seventh-grade students to examine the construction and consolidation process of Pythagorean Theorem by the help of GeoGebra. They stated that students could construct the expected knowledge and consolidate some of components that they had built. Kidron and Dreyfus (2014) carried out a study to exemplify the notion of proof image and to investigate how the proof images emerge. They used the abstraction in context as the theoretical framework and they stated that designing suitable tasks and providing opportunity to construct proofs can enrich students' mathematical experience and this is possible with RBC+C theory.

In this study, we used RBC+C model to reveal which knowledge occurred when they accomplished the postulates by paper folding and to observe how the teachers consolidated the aforementioned knowledge while constructing $\sqrt[3]{2}$. Mitchelmore and White (2007) tried to explain abstraction in mathematics learning. They emphasized two type of abstraction: (i) empirical and (ii) theoretical. And they stated that theoretical abstraction could be explained by RBC model. Further, they added that there is more than one theory for abstraction. Also they tried to distinguish between abstraction-from-actions and abstraction-from-objects in terms of theoretical approaches and stated that there was a need for more advanced theoretical work in research on mathematical learning and knowledge construction. Because of it, there is a need to relate RBC+C with another approach and in this study we prefer to relate RBC+C with MHM. "MHM" is a concept which looks into a person's own knowledge repertoire that enables him to overcome a problem (Goldenberg, 1996). According to this statement, it can be possible to reveal if there is a supportive relationship between MHM and RBC+C or not.

Mathematical Habits of Mind (MHM)

"MHM" was defined as preferring and using the suitable higher order cognitive skills among the others (Leiken, 2007). It helps a person about thinking mathematics as a way which was produced by mathematicians. It draws attention since determined characteristics of MHM refer to the characteristics of the nature of mathematics. In the literature, there isn't any standard list that reveals MHM (Lim, 2013). Habits of mind are dealt as general habits of mind for every discipline and as special habits of mind for mathematics by Cuoco, Goldenberg and Mark (1996). The most important point is that MHM and mathematical thinking are nested in each other. In other words, mathematical thinking includes MHM (Leikin, 2007). MHM means having a constant discernment by making thought experiments in non-routine situations, by taking into consideration the way which mathematicians follow and abstracting like them (Mark, Cuoco, Goldenberg & Sword, 2009). The characteristics of MHM make progress according to learning levels (Cuoco, Goldenberg & Mark, 2010; Cuoco & Levasseur, 2003). For mathematics, habits of minds specific to mathematics are dealt as algebraic habits of mind (Driscoll, 1999; 2001) and geometric habits of mind (Driscoll et al., 2007).

Algebraic habits of mind are doing/undoing (1), building rules to represent functions (2) and abstracting from computation (3) (Driscoll, 1999). Doing and undoing (1) involves reversing mathematical process, while building rules to represent functions (2) involves pattern-recognition and generalization. And abstracting from computation (3) involves thinking about computations structurally without specific situations (Lim & Selden, 2009). Doing-undoing (1) algebraic habit is considered as a roof component for the other two algebraic habits. This habit is always present in a problem solving case of the students. Building rules to represent functions (2) consists of pattern seeking, pattern recognition and generalization components, which in the analysis of problem solving process. Abstracting from computation (3), are the using structures and formulation of generalization about computation (Driscoll, 1999). Researchers have referred to algebraic habits of mind in different ways (Bass, 2008; Cuoco, Goldenberg and Mark, 1997; Matsuura, Sword, Piecham, Stevens, & Cuoco, 2013; Lim and Selden, 2009). However, the first systematic and in-depth representation were performed by

Cuoco,Goldenberg and Mark (1996). AHM allows working on the tasks and discussions used on the Algebraic thinking of the students in the classroom in a context.

Geometric habits of mind are a reproductive thinking way. The mentioned thinking way is investigating geometric relationships and reasoning with these relations (1), generalizing geometric ideas (2), investigating invariants (3) and balancing the exploration and reflection (4) (Driscoll et al., 2007). These four characteristics are geometric habits of mind.

According to Driscoll et al. (2007), Reasoning with relationships (1) is thinking regarding geometric figures, researching geometric relationships and using special reasoning skills. Generalizing geometric ideas (2) is understanding and identifying geometric phenomena. In this process, steps, results and characteristics of geometric figures are generalized. Investigating invariants (3) investigates the changing and unchanging situations and characteristics. Balancing exploration and reflection (4) tries different solutions in a problematic situation and returns previous steps to evaluate the situation continuously. In this study we emphasized both geometric and algebraic habits of mind and related these habits to each other with the abstraction process.

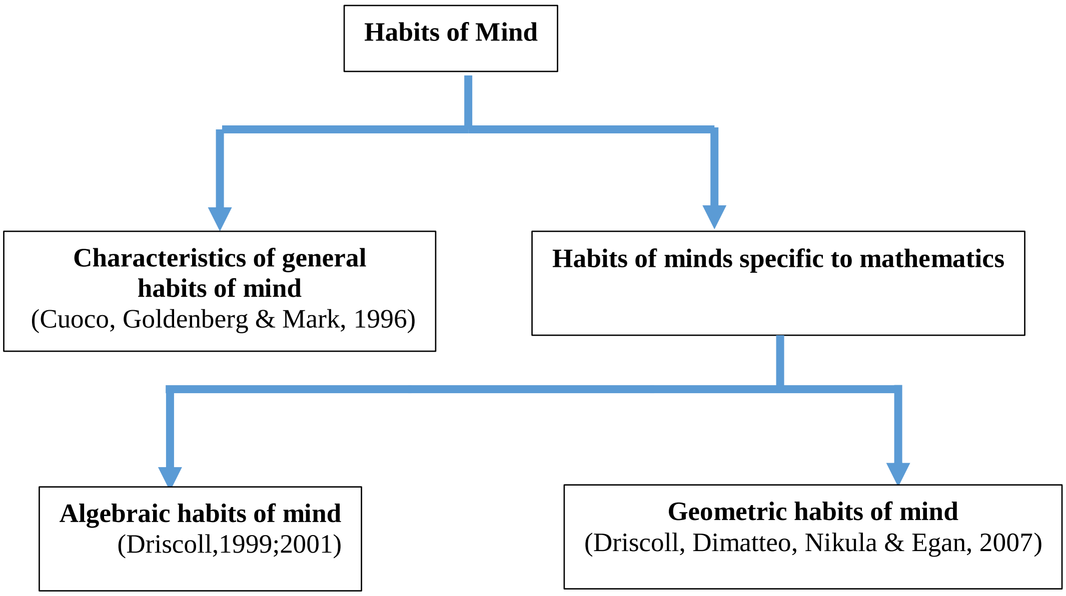

Figure 1. Habits of mind

The need for the mathematical habits of mind to help students think about the "path followed by mathematicians" has emerged (Lim and Selden, 2009). The main characteristics of the mathematical habits of mind are developmental stages (Cuoco, et. al. 2010; Goldenberg, et. al. 2003; Cuoco & Levasseur, 2003). On the other hand, advanced mathematical thinking (Leikin, 2007) is also seen as equivalent to mathematical power in supporting the learning and application of mathematics at the same time (Çimen, 2008). Mathematical power is defined by the best of all mental habits. Harel (2007, 2008) describes the direction of thinking of habits of mind with the concept of "ways of thinking" and emphasizes the internalization of ways of thinking as a habits of mind. The purpose of these habits is to help learners learn ways to think of problems and help them cope (Cuoco, et. al. 1996; Lim & Selden, 2009). Geometric Habits of Mind - GHOM contains geometry-specific components compared to the mathematical habits of the mind. Researchers working on this roof (Driscoll et al., 2007; Driscoll et al., 2008), consisting of four geometric habits associated with each other, stated that the framework they created was a perspective for geometric thinking. The structure of GHOM focuses on identifying evidence for geometric thinking (Driscoll et al., 2008; Koç & Bozkurt, 2012).

Driscoll et al. (2007) described the ways in which producers can think geometrically about how their 5^{th} - 10^{th} grade students can define geometric thinking. The sessions of the GHOM provide teachers with challenging mathematics problems and enables them to analyze artifacts of student thinking (Driscoll et al., 2007). They are encouraged to reflect on their understanding of geometry and nature of geometric thinking. The structured exploration process guides the activities in each of GHOM, to provide a meaningful cycle and explore and reflect on student thinking together (Driscoll et al., 2007).

Table 1. Geometrical Habits of Mind (GHoM) and indicators

Geometric Habits of Mind	GHoM Indicators	Student Indicators
1. Reasoning with relationships	Focuses on relations among separate figures	Determines the relationship between the properties of geometric shapes.
	Focuses on relations among the pieces in a single figure	Defines the properties of shapes – classification
	Uses special reasoning skills to focus on relations	Associates more than one geometric shape by proportional reasoning (parity - similarity)
2.Generalizing geometric ideas	Seeks solutions from familiar cases or known solutions	Generalizes by exploiting the exception to describe the problem state
	Seeks a range of solutions using assumed simplifying conditions	Adapts a general situation, including the problem, for the specific case
	Seeks complete solution sets or general rules	For the problem, you can think of all possible situations in the frame of the data.
3.Investigating invariants	Use dynamic thinking and searching	When a transformation is made geometrically, it determines which features of the shape have changed and which ones have remained fixed and solves the problem.
	Check evidence of effects	Geometric structure, it can be thought of as moving, explaining the effect that occurs, so as not to disturb the conditions of the problem.
4.Balancing exploration and reflection	Put exploration in foreground	You can make additional drawings to help solve the problem /It can develop different strategies to solve the problem.
	Put end goals in the foreground	Ask questions related to retro-metacognitive capacity in problem solving / Explain through mathematical language for correctness of problem solving.

Adapted from Driscoll et. al (2007

When analyzing the process by RBC+C model, it can be beneficial to use MHM to understand process well. There aren't many researches on MHM. Köse and Tanışlı (2014) tried to find out preservice primary teachers' geometric habits of mind. They stated that preservice teachers didn't have different geometric thinking ways and their habits weren't at the desired level. Leikin (2007) discussed multiple ways of problem solving as a habit of mind. According to her, solving problems in different ways is a MHM which requires and fosters advanced mathematical thinking. Jacobbe and Millman (2009) associated MHM with Polya's problem solving principles. They stated that rich problems would help preservice elementary teachers develop their MHM. As is seen, researchers generally related MHM to another approach.

Wiles (2013) stated that all the steps of the geometric habits of mind can be analyzed in detail by studying through a paper fold. Students fold paper to make and test conjectures while reasoning about and discussing geometric ideas. By focusing on geometric habits of mind, Wiles students not only have opportunities to explore important geometric ideas but also learn how to test ideas, make conjectures, pose new questions, and feel the thrill of uncovering relationships that appear, seemingly, out of nowhere. In this process, the paper folding exploration enriches tasks because they can motivate and interest students. From this point of view, the MHM was used to support the completion of RBC + C and the reflection of student responses in the explanation of the students' geometric thinking processes.

Paper folding (origami) is an art which is used for mathematics education nowadays, particularly for geometry. Noted educators, such as the German, Friedrich Froebel, have suggested the use of origami as a tool for the teaching of elementary geometric form (Geretschlager, 1995). Origami can be used to construct various geometric shapes. But it has postulates like Euclid Geometry. Thus, it needs to explain postulates of paper folding.

Postulates of Paper Folding

As Euclidean Geometry, paper folding has postulates and axioms. These postulate and axioms explain how the concepts as segment, angle, perpendicularity and congruence are defined in paper folding world and different researcher stated them in different ways (Auckly & Cleveland, 1995; Alperin, 2000; Geretschlager, 1995; Olson, 1975). However, the postulates which are known as Huzita's axioms are shorter, clearer and they are more useful for this study. According to Huzita the process of paper folding can be reduced to seven simple postulates (Krier, 2007).

Postulate 1: Given two points P1 and P2, one can fold a single crease which passes through them.
Postulate 2: Given two points P1 and P2, one can fold a crease placing P1 onto P2.
Postulate 3: Given two lines L1 and L2, one can fold a crease placing L1 onto L2.
Postulate 4: Given a point P1 and a line L1, one can fold a crease which will be $\perp$ to L1 and pass through P1.
Postulate 5: Given two points P1 and P2, and a line L1, one can fold a crease that places P1 onto L1 and passes through P2.
Postulate 6: Given two points P1 and P2 and two lines L1 and L2, one can fold a crease that places P1 onto L1 and P2 onto L2.
Postulate 7: Given a point P and two lines L1 and L2, one can fold a crease placing P onto L1 which is $\perp$ to L2.

We used only first six of them for this study, because it was enough to construct $\sqrt[3]{2}$. And we investigated the process in terms of RBC+C and MHM. Paper folding is a commonly used tool in geometry education. But it was usually used for the early grades to develop a positive attitude and to teach basic concepts of mathematics (Boaks, 2008; Johnson, 1957; Olson, 1975; Prigge, 1978; Polat, 2013). Some studies are about higher algebra and geometry (Auckly & Cleveland, 1995; Alperin, 2000; Geretschlager, 1995; Krier, 2007) and they aren't related to education. Hull and College (2007) showed how one could construct π by using paper folding. In this study we aimed to use paper folding to solve an impossible algebraic problem and construct the mentioned length ($\sqrt[3]{2}$) and to relate paper folding with education. Also Hull (1996) showed how one could construct $\sqrt[3]{2}$ and he stated that cubic equations could be constructed by paper folding.

Method

We worked with two high school mathematics teachers who were PhD students in the field of mathematics education. They were quite successful as doctoral students in university and teachers in their schools. Teacher 1 had 12 years' experience while Teacher 2 had 16 years. These teachers were open to new ideas and volunteer to participate in this study. The main reason of working with teachers was that it is important how teachers abstract the knowledge and what their mathematical habits of mind are and relation among these two situations. As stated before, these two (teachers' experiences and behavior) give shape to teaching method of teachers.

Teachers were asked to carry out the postulates and to construct $\sqrt[3]{2}$ at the end by paper folding. The experiment was carried out in a silent class and the participants were studied one by one in different days. They only used papers to fold and pencils to draw when they needed. They were given a paper on which the postulates were written, they were asked to accomplish the postulates respectively. As for 6th postulate; if they accomplish it, they will construct the desired length.

The data collection process was videotaped and it took nearly an hour for each teacher to complete. The collected data were transcribed and then they were analyzed by two researchers with descriptive methods, according to RBC+C, characteristics of consolidation and MHM. While constructing these lengths, the participants were expected to use algebraic expressions and in this process one could observe algebraic habits of mind and also the characteristics of consolidation (+C).

It was impossible to construct a cube with twice ($\sqrt[3]{2}$) by compass and ruler while it was possible by paper folding in various ways and one of which was Peter Messer's Solution (Krier, 2007). This solution was based on postulate 6. Therefore, we preferred to study on constructing $\sqrt[3]{2}$ and while the length was being constructed, the consolidation process of Pythagorean and Thales Theorem could be observed.

A priori Analysis and Expectations

In the first part of the study, the participants were asked to justify the postulates by paper folding. In the whole study, we used the six of paper folding postulates out of seven. Postulate 1 and 3 serve "reasoning with relationships" which is one of the geometric habits of mind. Postulate 2 serves "generalizing geometric ideas" which is the other one of the geometric habits of mind. Postulate 4 serves "balancing exploration and reflection" which is the another geometric habits of mind. The ultimate problem was to find a certain length by paper folding. When they tried to find this length, the consolidation process of Pythagorean and Thales Theorems could be clearly observed. Postulate 5 and 6 serve constructing length of $\sqrt{2}$ and $\sqrt[3]{2}$. The expectations from participants can be seen below.

Postulate 1: Given two points P1 and P2, one can fold a single crease which passes through them.

They should be asked to explain what they have constructed and whether they can construct another crease passing through these two points. We expect them to say that the crease is a line and there is one and only one line. According to RBC+C theorem, if they say this crease is a line, we can say that they recognize the line and line is the knowledge which has already been consolidated. In terms of geometric habits of mind, if they recognize that there is only one line because of Euclid's postulates, we can say that they associated the relation between paper folding postulates and Euclid's.

Postulate 2: Given two points P1 and P2, one can fold a crease placing P1 onto P2.

They should be asked to explain what they have constructed by the help of postulate 1 and 2. The expected answer is that they can construct perpendicular lines. At that point, the important question is "why?". The expected reason is that all the angles are congruent. According to RBC+C, we can say that they recognize the perpendicularity knowledge which has been consolidated already. In terms of geometric habits of mind if they recognize the reason of perpendicularity, we can say that they generalize geometric ideas.

Postulate 3: Given two lines L1 and L2, one can fold a crease placing L1 onto L2.

They should be asked what they have constructed and what they can say about the crease. We expect them to say that the crease is an angle bisector. According to RBC+C theorem if they say this crease is an angle bisector, we can say that they recognize the angle bisector and angle bisector knowledge has already been consolidated. In terms of geometric habits of mind, for the same answer, we can say that they explain why it is angle bisector and relation between paper folding and the crease.

Postulate 4: Given a point P1 and a line L1, one can fold a crease which will be $\perp$ to L1 and pass through P1.

If the participants construct the Postulate 4, we can say that the perpendicularity knowledge has already been consolidated. Because, Postulate 4 is a different type of Postulate from 1 & 2. If they explain why these lines are perpendicular, this situation refers to habit "Balancing exploration and reflection". The only reason for asking to construct postulate 5 and 6 is to use them when constructing the lengths of $\sqrt{2}$ and $\sqrt[3]{2}$. The process can be seen as follows.

Postulate 5: Given two points P1 and P2, and a line L1, one can fold a crease that places P1 onto L1 and passes through P2.

The folding of Postulate 5 is as follows:

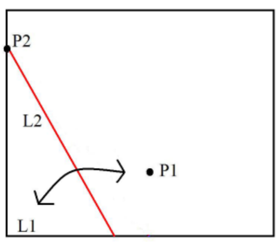

Figure 2. Perform postulate 5 (Krier, 2007)

They should be asked to construct postulate 5 in order to use it when they construct √2 by paper folding. For this purpose we give them a sheet of paper with special characteristics. The paper is a square which is divided into three equal parts. And we ask them to perform postulate 5 with this paper and after that to construct √2. The figure is as follows:

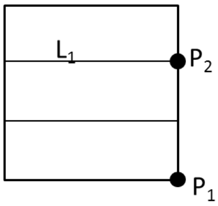

Figure 3. Perform postulate 5 with a special paper

By constructing √2 with the help of folding, we expect them to use similarity and Pythagorean Theorem. With this activity, we expect to see if they recognize these knowledges or not. In this process, they should write algebraic expressions. For this reason, the process is analyzed in terms of algebraic habits of mind.

Postulate 6: Given two points P1 and P2 and two lines L1 and L2, one can fold a crease that places P1 onto L1 and P2 onto L2.

They should be asked to construct postulate 6 to use it while constructing $\sqrt[3]{2}$ by paper folding. By constructing $\sqrt[3]{2}$ with the help of folding, we expect them to use similarity and Pythagorean Theorem. With this activity, we expect to see if they consolidate these knowledges. Because they use them to construct √2 and we would like to see if they can use it for another situation. If they do so, we can say that they consolidate similarity and Pythagorean Theorem knowledge. After constructing these postulates by paper folding, we asked them if y=1; whether they could calculate X, which is the below mentioned length. This paper is a square and divided into three equal parts, like postulate 5.

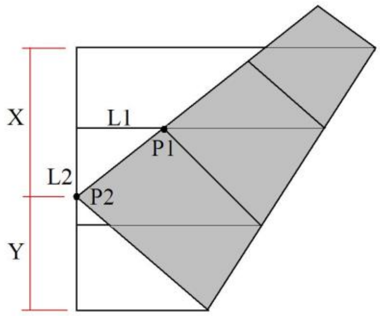

Figure 4. Perform postulate 6 (Krier, 2007)

We expect them to find the lengths like below the figure and do algebraic operations to find X, which is $\sqrt[3]{2}$.

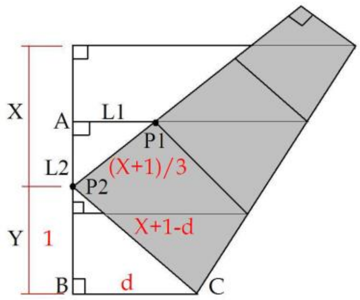

Figure 5. The Needed Lengths (Krier, 2007)

Let BC = d. Since the bottom edge equals X+Y = X + 1, this results in P2C +d = X + 1, this results in P2C=X+1- d. Rewriting d via the Pythagorean Theorem we get,

$$d = \frac{x^2 + 2x}{2x + 2}$$

Also, notice that P1P2= $\frac{1}{3}$s, which in terms of X is $\frac{x+1}{3}$. We can also derive the value of AP2 by taking X and subtracting $\frac{1}{3}$s, giving us a value of $\frac{2x-1}{3}$. Now, by Haga's Theorem which remarks three similar triangle, we know that ΔP2AP1 is similar to ΔCBP2. Therefore we can say;

$$\frac{d}{x+1-d} = \frac{2x-1}{x+1}$$

$$\frac{x^2+2x}{x^2+2x+2} = \frac{2x-1}{x+1}$$

$$x^3+3x^2+2x = 2x^3+3x^2+2x-2$$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$ (Krier, 2007)

Results

Findings were stated in tables below. Also general ambiance of study is explained behind tables for each postulate and each participant. We asked them to fold the paper as P1 and P2. The results can be seen in Table 2.

Table 2. Postulate 1 & 2

P.	Participant	RBC+C	Reference	Habits of mind	Reference
1	Teacher 1	Recognizing	Statement: There is only one line.	Reasoning with relationships	Before researcher asked to construct a crease he said “only one line”. He associated Postulate with Euclid’s Postulate.
	Teacher 2	Recognizing	Statement: Only one line passes through given two points.	Reasoning with relationships	After folding, he associated the postulate with Euclid’s.
2	Teacher 1	Recognizing	After folding, he directly said “The lines intersect each other at perpendicular”	Generalize geometric ideas	He explained the perpendicularity with different references. First, he said that there was an isosceles triangle and angle bisector is the perpendicular bisector at the same time. Second explanation was “When we fold, this point is a vertex of the rectangle. Therefore it must be perpendicular”
		Consolidation (+C)	It can be seen that he explains why the lines are perpendicular with the different references. So this refers to the flexibility which is a characteristic of consolidation.		
	Teacher 2	Recognizing	After folding, he directly said “The lines are perpendicular.”	Generalize geometric ideas	He said “When I fold the paper like this, symmetry axes of a rectangle were formed. Because of that they are perpendicular.”

Also, they could accomplish the P3 easily. When asked to explain what they did, Teacher 1 could easily say the set of the points which were located the same distance from two lines while Teacher 2 said directly it was a symmetry axis. Then when asked if it had another name, both of them said that at the same time it was angle bisector. When they carried out P4, both of them didn’t have difficulty. They could explain that the postulate was similar with combination of P1 & 2.

Table 3. Postulate 3 & 4

P.	Participant	RBC+C	Reference	Habits of mind	Reference
3	Teacher 1	Recognizing Consolidation	Statement: The line is the set of points which are located same distance from these two lines. Also angle bisector.	Reasoning with relationships	He could explain why it was an angle bisector.
	Teacher 2	Recognizing Consolidation (+C, flexibility)	Statement: This is the symmetry axis of them exactly. Also, it can be angle bisector. Because angle bisector is that the line is the set of points which are located same distance from these two lines.	Reasoning with relationships	He could explain why it was an angle bisector and symmetry axis and relation between these two concepts.
4	Teacher 1	Consolidation (+C)	Statement: It is perpendicular because I constructed a rectangle and this point is vertex of the rectangle so it is perpendicular to L1.	Reasoning with relationships	He associated Postulate 4 with the vertex of rectangle as well as he did in postulate 2. He didn't need to justify the truth of perpendicularity.
	Teacher 2	Consolidation (+C)	He folded the paper but he wasn't able to fold the crease as perpendicular. But he persistently said it had to be perpendicular. And then he folded the paper again properly.	Balancing exploration and reflection	He associated Postulate 4 with postulate 1 and 2 (He constructed perpendicular lines). He knew the crease was perpendicular but by using these two postulates, he tried to prove its truth. He justified the perpendicularity.

P5 and P6 were a little bit difficult for the participants. Teacher 1 read P5 again and again and tried to understand and imagine what he was asked. He said that it wasn't possible for any points instead of particular cases. After that, we asked him if he was sure, he tried to do and accomplish P5. Then he was given a particular case (which was prepared by the researcher before.) and he was asked whether he could obtain the measure of $\sqrt{2}$. He answered immediately and said that he could use Pythagorean Theorem and the similarity of triangles and he obtained $\sqrt{2}$ by folding. He explained how and why he used these two knowledges.

Teacher 2 had some difficulty to understand what P5 meant. He said that he understood the postulate as a reflection of a point with respect to a line and he folded properly to this idea. Then he understood P5 truly, he started to think about how he could fold. He accomplished P5; but it took a while. He could fold the sheet, which the researcher gave as a particular case. Then he obtained $\sqrt{2}$ without any doubt by folding and using Pythagorean Theorem and the similarity of triangle.

For P6, Teacher 1 wanted to think a while. When he couldn't fold, he preferred to fold the particular paper which we had prepared before. He folded the sheet easily. But he needed to generalize about this postulate "How and why can I fold the sheet like that?"

Table 4. Postulate 5

P.	Participant	RBC+C	Reference	Habits of mind	Reference
5	Teacher 1	Consolidation (+C)	When length of √2 was asked, he directly said "I use similarity." He explained how he could use the Thales theorem. And he said the hypotenuse of triangle is √2. When asked the why, he directly said because of Pythagorean Theorem.	Algebraic habits of mind Balancing exploration and reflection	He calculated the lengths systematically. This referred to "abstracting from computation" which was one of algebraic habits of mind. Statement: I am trying to imagine the construction of the desired folding… Where should I choose the points? Which one is easier?... These statements referred to "balancing exploration and reflection".
	Teacher 2	Consolidation (+C)	When length of √2 is asked, he directly said this length is √2. When asked the reason, he said because of Pythagorean Theorem. Then he explained how he found the length and said "If we refer to similarity, the ratio is ½."	Algebraic habits of mind Balancing exploration and reflection	He calculated the length by his mind. He didn't need to write. And he explained the operations systematically. So it can be said that these actions refer to abstracting from computation. Statement: I am going to think a little bit then I will tell how I can fold (He tried to imagine folding). When he started to fold paper he said "Ok then, I can construct a line passing through this point… But what is the relationship between these two lines (It can be seen that he tries to justify folding and construction).

After that the researcher asked him to obtain $\sqrt[3]{2}$, he repeated what he knew. He constructed similar triangles by folding but he had difficulty in equating. The equations which he wrote were with two unknowns and higher degree equations so he thought he couldn't solve them. He thought about how he could write 'a' like 'x'. He continued to use Pythagorean and Thales Theorem. When he obtained $\sqrt[3]{2}$, he was shocked. He didn't believe that he could write 'a' like 'x' and obtain the desired measure. Teacher 1's operations are as follows:

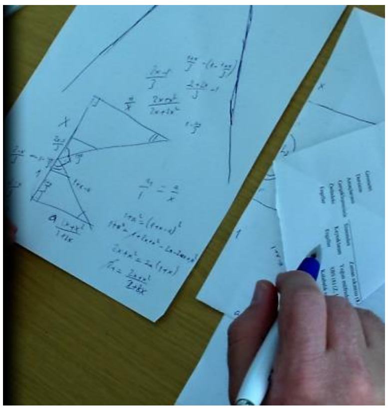
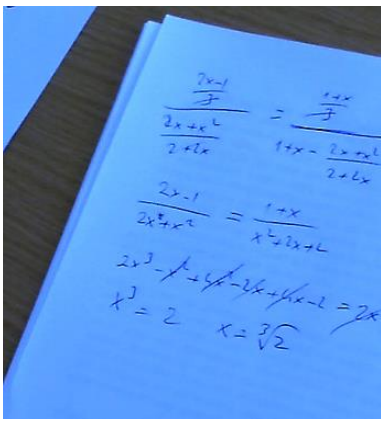

Figure 6. Teacher 1's operation

Table 5. Postulate 6 (Teacher 1)

P.	Participant	RBC+C	Reference	Habits of mind	Reference
6	Teacher 1	Consolidation (+C)	When length of $\sqrt[3]{2}$ was asked, he said "Similarity can be used with these two triangles."(immediacy) Then he suggested constructing new triangles with a new folding (flexibility). While he was calculating the length, he used similarity and Pythagorean Theorem. Even if he couldn't calculate, he didn't give up using these knowledges and tried to do by using different ways but again with these knowledges. (awareness, confidence, self-evidence, flexibility) .	Algebraic habits of mind	Sometimes he made operations with his mind and sometimes he wrote algebraic expressions. He used variables "a, x, d, y" and wrote equations to calculate the desired length. He had a belief that he couldn't find the 'x'. But at the end of the study he could.
				Balancing exploration and reflection	He tried to fold postulate 6 properly. He said "I try to imagine the folding, if the line goes along how it will be."
				Generalize geometric ideas	He couldn't fold any crease as general but he could fold a crease with the given special paper which was divided into 3 equal parts. And he said "I couldn't understand why I could accomplish this special situation and I couldn't generalize." It can be seen from these expressions that he needed to generalize the folding.

Teacher 2 tried to fold the sheet in some way and then determine the suitable point and line. He tried to ignore one of the conditions. This process reminded of Polya's problem solving principles. He folded it for a particular case then he folded the sheet which we gave. After several trials, he accomplished P6. When we asked him to obtain$\sqrt[3]{2}$, he said immediately that he could use the similarity knowledge. He tried to find the desired measure but he didn't believe that he could obtain. He continuously used Pythagorean and Thales Theorem. He had difficulty in solving the equations. He didn't believe that he could find 'x' from these equations. Then he decided to use Pascal's triangle.

Table 6. Postulate 6 (Teacher 2)

P.	Participant	RBC+C	Reference	Habits of mind	Reference
6	Teacher 2	Consolidation (+C)	When length of $\sqrt{2}$ is asked, he directly said this length is $\sqrt{2}$. When asked the reason, he said because of Pythagorean Theorem. Then he explained how he found the length and said "If we refer to similarity, the ratio is ½."	Algebraic habits of mind	He calculated the length by his mind. He didn't need to write. And he explained the operations systematically. So it can be said these actions refer to abstracting from computation.
				Balancing exploration and reflection	Statement: I am going to think a little bit then I will tell how I can fold (He tried to imagine folding). When he started to fold paper he said "Ok then, I can construct a line passing through this point… But what is the relation between these two lines? (It can be seen that he tries to justify folding and construction).

Conclusion and Discussion

Constructing $\sqrt[3]{2}$ is a well-known ancient problem and its solution is possible using paper folding. It is an important issue for teachers and researchers; it is because this can prompt them to construct other cubic roots by using paper folding. Hull (1996) stated that cubic equations could be constructed by using paper folding. From the point of this view, cubic equations can be used for constructing general cubic roots, not only $\sqrt[3]{2}$. But the first step should be constructing $\sqrt[3]{2}$; it is because constructing $\sqrt[3]{2}$ by paper folding hasn't already known among teachers and also researchers.

The main purpose of this study was to investigate mathematics teachers' mathematical thinking process while they were constructing the length of $\sqrt[3]{2}$ by paper folding. To investigate the mentioned process, it was studied along with two teachers who were PhD students. The sub-goals are "to analyze construction process of $\sqrt[3]{2}$ in terms of RBC+C model and MHM", "to analyze the consolidation process of Pythagorean Theorem and Thales Theorem" and "to determine whether these two approaches (RBC+C model and MHM) corroborate each other or not during the abstraction process".

As Monhagan and Ozmantar (2006) and Dreyfus (2012) determined, it was seen that a construct could be consolidated when it was used in further activities. It was determined with P5 whether the teachers had constructed knowledge of Pythagorean and Thales Theorems and with P6 it was observed whether these theorems were consolidated or not. As Dreyfus (2012), Anabousy and Tabach (2015) stated, consolidation is a never-ending process. Even if they are teachers, they need to improve their constructions in their mind. When considered from the point of the mentioned theorems, it can be said that both two teachers have knowledge of these theorems and also one can observe that this knowledge has been consolidated. But only in terms of "confidence" characteristic, they need to improve themselves. Because, especially during the accomplishing P6, they sometimes stopped and they couldn't be sure whether they were on the right direction or not. Correspondingly with Jacobbe and Millman's (2009) statements, MHM is related to problem solving principles. While accomplishing P6, teachers acted like that they were solving a problem and their actions referred to Polya's problem solving principles which are (i) understand the problem, (ii) devise a plan, (iii) carry out the plan and (iv) look back. And this situation corroborates the thought (Leikin, 2007) of MHM that regards mathematical thinking process.

In this study, the focus is on analyzing mathematical thinking during the construction process rather than obtaining$\sqrt[3]{2}$. On the one hand, the construction process can be analyzed by RBC+C model and on the other hand, it is possible to analyze it in terms of MHM. Likewise Leikin (2007), it was found that MHM related to mathematical thinking. In addition, MHM served to mathematical thinking and these two approaches (RBC+C and MHM) were close and corroborated each other. And more remarkable issue is that especially consolidation process corroborates MHM. An also, as stated by Wiles (2013), MHM, especially GHOM (geometric habits of mind) can serve the foundation mathematical ideas by paper folding. It can be seen in the findings. Teacher's mathematical thinking process and habits of mind can be revealed by the help of paper folding.

It is clearly seen from P1 and P3, flexibility, which is one of the characteristics of consolidation, is related to "generalize geometric ideas" and "reasoning with relations". Their reference sentences are similar. Considering P6, it can be clearly seen that all the characteristics of consolidation occurred. And in terms of MHM, geometrical and algebraic habits of mind were observed."Balancing exploration and reflection" is an important habit and it can be seen that teachers had this habit. Also emerging algebraic habits of mind is a result of flexibility. Because, as an example, Teacher 2 would like to use different notations and solutions while he solved the equations. This refers to both consolidation and algebraic habits of mind.

Herkowitz et al. (2014) emphasized DCA and RBC+C are different methodologies but closely related to the classroom learning process. In parallel with their study, this study showed that RBC+C can be used with MHM which is different from RBC+C model but related with it. In addition, it can be possible to say that RBC+C model can be used with other approaches to analyze the process of mathematical thinking and abstraction more clearly.

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