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Abstract

This study has aimed to determine 8th grade middle school students' skills and knowledge on irrational numbers. A total of ten 8th grade students from a public school in Polatlı District of Ankara were participated. In the study, a basic qualitative research design was used, and data were collected by means of clinical interviews. Interview questions were developed based on a review of the relevant literature, and the final versions of the questions were determined in accordance with expert opinions. Analyses of the resultant data revealed that students experienced some difficulties when defining irrational numbers. They were relatively good at classifying numbers as irrational or not and explaining the reason for this classification. Nevertheless, it was concluded that they had difficulty identifying the relationships among sets of numbers, explaining the difference between rational and irrational numbers and performing operations with irrational/rational numbers. A set of recommendations were developed for overcoming those challenges. Among them were using different types of representations during the instruction process, making use of non-routine problems, making connections to the history of mathematics and using calculators.

Introduction

Humans deal with numbers when performing operations such as measuring and counting in order to solve the problems that occur in their daily lives (Balcı, 2008; Sirotic, 2004). Real numbers are among the subjects through which mathematics is connected to the real world. The set of real numbers, the largest known set of numbers, consists of a combination of the sets of rational and irrational numbers. The most widely used definition of the set of rational numbers is *"a ratio of two integers"* while the set of irrational numbers is defined as "*all the real numbers which are not rational, in other words, "numbers that cannot be put in the form a/b where $a \in Z$, $b \in Z$, and $b \neq 0$.*" in books (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014, p. 229). Irrational numbers are described via this formal definition, yet, they are also known as *"numbers which do not have a repeating or terminating decimal expansion"* (OCG, 2005). Throughout the history, it has not been easy to identify the distinctive features of irrational numbers. Ancient people believed it was possible to write all numbers as a ratio of integers; however, they made one of the greatest discoveries in human history when they found out that results of some calculations could not be expressed using the then known numbers (Sertöz, 2011). In the following times, Dedekind, Cantor and Weierstrass filled a great gap in the literature thanks to the theories they developed (Fischbein, Jehiam, & Cohen, 1995). Making sense of irrational numbers within the scope of the set of real numbers entails a complex process just as its discovery in the history of mathematics was. Apart from their complex nature, it should be noted that irrational numbers are key to understanding advanced mathematics (Güven, Çekmez & Karataş, 2011). Furthermore, the number π, which is widely used in mathematics, and mathematical engineering, Euler's number e (2.7182818…), and the golden ratio that we come across both in our daily lives and the nature (1.6180339887…) are all special irrational numbers.

A review of the mathematics curricula in Turkey reveals that students are introduced irrational numbers in the eighth grade (MoNE, 2018a). In high school and university years, students build on their subject knowledge by studying subjects that are based on these numbers. An analysis of the skills and knowledge that the curriculum aim to deliver demonstrates that two main objectives are emphasized (MoNE, 2009). The first one is to explain the difference between rational and irrational numbers. In this regard, discussing whether all decimal expansions can be written in rational numbers' form or not is regarded as a requirement. It is considered necessary to explain by means of examples that rational numbers can be expressed as a ratio of two integers (denominator not equal to zero), but irrational numbers cannot be written as a ratio of two integers. Some explanations are provided through examples on repeating decimal expansions. The second point of emphasis, on the other hand, is identifying the sets of numbers that make up the set of real numbers. Besides, there are some activity suggestions for teaching students how to locate the point to which a real number corresponds on the number line (MoNE, 2009). In the revised mathematics curriculum, in the eighth grade, the relationship between real numbers, and rational and irrational numbers is underlined. Additionally, it is emphasized that irrational numbers cannot be written in the form of a ratio of two integers (MoNE, 2018a). It is seen that the relationship between sets of numbers and the points to which irrational numbers correspond on the number line are addressed in the ninth grade (MoNE, 2018b).

Students are offered real world models and representations about natural numbers, integers and rational numbers before irrational numbers are taught; however, they cannot be offered such representations or real world models of irrational numbers, since these numbers are more abstract by nature. As a consequence, students experience a variety of difficulties regarding irrational numbers (Güven, et al., 2011). Irrational numbers are essential for expressing numbers and making sense of irrationals involves quite a complex process. As a result, this has urged researchers to conduct more research on the subject. A review of the relevant literature shows that several studies have been carried out regarding the subject. It has been understood that those studies have mostly been conducted with in-service and pre-service teachers (Arbour, 2012; Arcavi, Bruckheimer, & Ben-Zvi, 1987; Baştürk & Dönmez, 2008; Çiftçi, Akgün, & Soylu, 2015; Fischbein, Jehiam, & Cohen, 1995; Güler, 2017; Güler, Kar, & Işık, 2012, Güven, et al., 2011; Kara & Delice, 2012; Mamolo, 2009; Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis 2007a, 2007b; Zazkis & Sirotic, 2004). There are also studies that have focused on high school (Ercire, 2014; Fischbein, et al., 1995; Kara & Delice, 2011) and middle school students (Adıgüzel, 2013; Ercire, 2014; Ercire, Narlı, & Aksoy, 2016; Temel & Eroğlu, 2014).

When the content of those studies is reviewed, it is seen that each focused on a different dimension about irrational numbers. These are the definition of irrational numbers (Arbour, 2012; Arcavi, et al., 1987; Baştürk & Dönmez, 2008; Ercire, 2014; Ercire, et al., 2016; Fischbein, et al., 1995; Güven, et al., 2011; Güler, 2017; Kara & Delice, 2012; Sirotic, 2004), classification of given numbers as rational or irrational (Adıgüzel, 2013; Arbour, 2012; Çiftçi, et al., 2015; Ercire, 2014; Fischbein, et al., 1995; Güven, et al., 2011; Kara & Delice, 2012; Peled & Hershkovitz, 1999; Sirotic, 2004; Temel & Eroğlu, 2014), different representations of irrational numbers (Peled & Hershkovitz, 1999; Sirotic, 2004; Zazkis & Sirotic, 2004), the places of irrational numbers on the number line (Güler, 2017; Güven, et al., 2011; Peled & Hershkovitz, 1999; Mamolo, 2009; Sirotic, 2004; Sirotic & Zazkis, 2007a, 2007b), operations that involve irrational numbers (Adıgüzel, 2013; Arbour, 2012; Ercire, 2014; Güler, 2017; Güler, et al., 2012, Güven, et al., 2011; Sirotic & Zazkis, 2007a) and relationships between sets of numbers (Adıgüzel, 2013; Arbour, 2012; Ercire, et al., 2016; Kara & Delice, 2012).

Studies on how irrational numbers are defined were analyzed, and it was concluded that participants of the subject studies failed to explain the concepts of rational and irrational number. In addition to the difficulties they experienced when defining those concepts, the subject participants were understood to suggest definitions which were far from being formal and which were based on representations and intuitions. It was clarified that their definitions of irrational numbers mostly entailed such statements as “numbers that are not rational” or “numbers that have infinite and non-periodic decimal representations” (Arbour, 2012; Arcavi, et al.,, 1987; Baştürk & Dönmez, 2008; Ercire, 2014; Ercire, et al., 2016; Fischbein, et al., 1995; Güven, et al., 2011; Güler, 2017; Kara & Delice, 2012; Sirotic, 2004).

In the past works that studied numbers according to whether they were rational or irrational, participants were found to have difficulty categorizing numbers as rational or irrational. Besides, participants’ explanations for their statements revealed their lack of knowledge. Participants who used statements such as “the number π is rational because it equals to 22/7, repeating numbers are irrational as they are infinite” experienced difficulty determining which set of numbers the given numbers belonged to. Although they were able to identify some numbers as irrational, they could not explain the reason for this (Adıgüzel, 2013; Arbour, 2012; Çiftçi, et al.,, 2015; Ercire, 2014; Fischbein, et al., 1995; Güven, et al., 2011; Kara & Delice, 2012; Peled & Hershkovitz, 1999; Sirotic, 2004; Temel & Eroğlu, 2014).

Studies on different representations of irrational numbers demonstrated that pre-service teachers experienced difficulties in activities that involved different representations although they knew both the definition and properties of irrational numbers. In addition, some inconsistencies were revealed in their knowledge on the representation of irrational numbers as decimals and as fractions. They disregarded the fact that the possibility of using the expression a/b helps one tell irrational and rational numbers apart. Instead, they usually tended to use decimal representations (Peled & Hershkovitz, 1999; Sirotic, 2004; Zazkis & Sirotic, 2004).

Studies that explored knowledge on locating irrational numbers on the number line concluded that students had difficulty when spotting the point a given rational number corresponded to on the number line. It was also challenging for the participant students to associate irrational numbers with the concept of geometric length. Besides, they could not understand that those numbers are expressed as points on the number line, and they tended to indicate those points based on intuitions (Güler, 2017; Güven, et al., 2011; Peled & Hershkovitz, 1999; Mamolo, 2009; Peled & Hershkovitz, 1999; Sirotic, 2004; Sirotic & Zazkis, 2007a, 2007b).

A review of the studies on operations with irrational numbers showed that participants of the subject studies could not conceive that each irrational number is actually a real number. Accordingly, they had conceptual difficulties as to what those numbers meant in the given operations (Adıgüzel, 2013; Arbour, 2012; Ercire, 2014; Güler, 2017; Güler, et al., 2012, Güven, et al., 2011; Sirotic & Zazkis, 2007a). Finally, studies that focused on the relationships between sets of numbers revealed that participants could not conceive irrational numbers as real numbers, and as a result, had difficulty placing irrational numbers under the set of real numbers. Instead, they regarded rational numbers as a subset of irrational numbers and thus, found it hard to grasp the relationship between the two (Adıgüzel, 2013; Arbour, 2012; Ercire, et al., 2016; Kara & Delice, 2012).

A collective review of all the studies demonstrated that in- and pre-service teachers as well as high and middle school students experienced a variety of difficulties regarding irrational numbers. It was understood that almost 1/3 of the eighth graders were able to make a formal definition of irrational numbers and associated them with radicals or decimal places (Ercire, 2014; Ercire, et al., 2016). They tended to classify not only numbers with non-terminating decimal expansions but also radical numbers as irrational numbers, yet failed to explain the reason for doing so (Adıgüzel; 2013; Ercire, 2014; Temel & Eroğlu, 2014). In studies on operations with irrational numbers, it was established that it was hard for students to understand operations with irrational numbers (Ercire, 2014). It was shown that 8th grade students could not conceive irrational numbers as real numbers, and their knowledge was, in some regards, merely memorized information (e.g. the sum of a rational number and an irrational number is a real number) (Adıgüzel, 2013; Ercire, et al., 2016). When the past studies were analyzed, it was noteworthy that the number of accessible studies which focused on 8th graders was quite low in the relevant literature. Besides, those studies addressed 8th grade students together with 9th grade students and teachers (Adıgüzel, 2013; Ercire, 2014; Ercire, et al., 2016). Accordingly, it emerged as a necessity to explore the ways of thinking of 8th grade students as this is the grade when students are introduced the set of irrational numbers. In this regard, learning how those students think is also important for designing the learning environment to obtain higher productivity. In the relevant literature, the number of studies that handled several aspects together (defining and differentiating irrational numbers, the relationships between sets of numbers and the difference between them, operations with those numbers) and explored them comprehensively was limited. On the grounds of these, the present study explored students' skills and knowledge concerning the definition of irrational numbers, classification of a given set of numbers as irrational or not, the relationships between sets of numbers, the difference between rational and irrational numbers, and operations with irrational numbers. It is believed that the study will provide clues as to how learning environments can be more productive for the instruction of irrational numbers. For this purpose, answers to the following questions were explored:

- ✓ How do students define irrational numbers?
- ✓ How do students explain the relationships between sets of numbers?
- ✓ How do students classify numbers as irrational or rational?
- ✓ How do students explain the difference between rational numbers and irrational numbers?
- ✓ How do students determine whether the results of operations with rational and irrational numbers are irrational or rational?

Method

In this study, a basic qualitative research design was used. The aim of this method is to reveal and interpret the meaning of the individual's understanding. By this way, important clues can be achieved about how people conceptualize the focused situation and how they construct it in their minds (Merriam, 2013). In this study, a basic qualitative research methodology was adopted because the present study aimed to explore how eight grade students defined and identified irrational numbers, and how they associated them with other numbers and rational numbers.

Participants

A total of ten 8th grade students who received mathematics courses in accordance with the middle school curriculum of 2009 at a public school in Polatlı District of Ankara participated in the study. The reason why eighth grade students was participated that they encountered this subject firstly in this grade. In order to get the knowledge of the irrational numbers in detail, 11 students whose math grades were above the average of the class, were selected in the direction of teacher opinions and math grades. Five of the students were female and six were male. Names of the participants students were not disclosed, and they each were assigned codes such as S1, S2 …S10. The age of the students were ranged between 13-14. It was determined that the families of the students generally migrated from other cities and were socioeconomically at the middle level.

Data Collection Tools and Data Analysis

In the study, clinical interviews were conducted to explore how 8th grade students defined and identified irrational numbers, and how they associated them with other numbers. Clinical interviews help researchers get detailed information about what students know, and how they think (Clement, 2000). They also offer researchers the opportunity to explain the solution strategies they prefer without restricting the participants. As a consequence, clinical interviews enable researchers to discover students' ways of thinking in a more comprehensive way (Hunting, 1997). The present study aimed to explore and reveal students' knowledge and to obtain some clues regarding how they thought. The clinical interview allows students to be asked questions that reveal how they think during and after their answers. In this way, their thinking processes can be observed better. For this purpose, mathematics curriculum was analyzed. Following this, what the students were expected to know was identified. On the basis of these, an interview form of 5 questions was prepared. The interview form was developed using the studies by Fischbein et al., (1995) and Güven et al., (2011). The final version of the form, on the other hand, was determined in accordance with expert opinions. The aim in doing so was to ensure internal consistency. A pilot study was conducted with a student in order to find out whether the questions were clear and comprehensible and to analyze to what extent the questions could elicit the anticipated answers. The final version was ready after the necessary amendments were made. Clinical ınterviews aim to describe the internal world of an individual in detail and to reveal comments, opinions and mental perceptions that are otherwise invisible (Yıldırım & Şimşek, 2013). Interviews with the students lasted 40 to 60 minutes. The table below presents the interview questions and the categories of these questions.

Table 1. Questions and the categories of the questions

Category	Questions
Defining Irrational Numbers	What do you think when you hear the word "irrational numbers"? How do you define these numbers? Can you give an example to these numbers?
Classifying given numbers as irrational or rational	Can you classify the numbers given below as irrational or not? How would you explain your answers? 0.16, $13.\overline{2}$, -2.010101…, 14.01010304405…., √9, 2√3; 22/7, π
Identifying the relationships between sets of numbers	If you were asked to represent the sets of real numbers, rational numbers, integers, natural numbers and irrational numbers in the form of a diagram, what kind of a representation would you provide?
Telling the difference between rational and irrational numbers	Assume that somebody asks you to explain the difference between rational and irrational numbers. How would you explain the subject person this difference?
Determining whether the results of operations with rational and irrational numbers are irrational or rational	Please mark the following statements as true or false and indicate the reasons for your response. The sum of any two irrational numbers will always be an irrational number. The sum of any two rational numbers will always be a rational number. Multiplying one irrational number by another must produce an irrational result. Multiplying one rational number by another must produce a rational result.

In the interviews, students were expected to offer their own solutions, and they were allowed to think aloud while they were trying to find the solution. The resultant data were analyzed via content analysis method. After the codes for the data were defined, they were evaluated with a professional mathematics educator in the area. Interview data were analyzed one by one for each question. Responses were evaluated in two steps. Firstly, student responses were coded based on the content of their statements. Following this, those responses were labeled as correct (C), incomplete (IC), incorrect (I) or no explanation (NE), and those labels were given in parentheses. After these analyses, intercoder reliability was examined with a mathematics educator, and the codes were assessed and discussed during this process. Codes on which there was disagreement were revised and discussed until agreement was built. The resultant data were checked for reliability, and the formula "Reliability= Agreements / Agreements + Disagreements" was used for this purpose (Miles and Huberman, 1994). Intercoder agreement rate for the two coders was calculated to be 95%.

Findings

In this section, findings were assessed within the framework of the research questions, which explored students' ability to define irrational numbers, classify given numbers as irrational or not, tell the difference between rational and irrational numbers, identify the relationships between sets of numbers and perform operations with irrational numbers. Findings for the first research question are presented below.

"How Do Students Define Irrational Numbers?"

This research question required students to define irrational numbers. The statement "*numbers that cannot be written as a ratio of two integers are defined as irrational numbers,*" which is provided in course books, was taken into consideration during the analysis of students' responses to the subject question. Student responses that involved this statement were considered correct. Responses which explained certain properties of irrational numbers (digits after the decimal point do not form a periodic pattern, decimal part never terminates, etc.), yet did not provide the expected definition were marked as incomplete. Student responses that were not accurate, on the other hand, were marked as incorrect. Correctness, incorrectness or incompleteness of the resultant codes is presented in the table below.

Table 2. Students' definitions of irrational numbers

Codes	Student
A number that is not rational. (C)	S1, S3, S4
A number that cannot be written in the form of a fraction. (C)	S5, S7
Numbers with digits which do not have a periodic pattern after the decimal point. (IC)	S8, S9
Numbers with decimal parts that continue to infinity. (IC)	S2, S6
Repeating numbers. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

S1, S3 and S4 defined irrational numbers as numbers that are not rational. When they were asked to explain their responses through an example, S3 could not give any examples while S4 and S1 gave correct examples. Below is an excerpt from the interview with S4:

> "……………
> S4: When I hear the word irrational numbers, I think of numbers that are not rational.
> Researcher: In what other words would you describe those numbers or do you want to give an example?
> S4: I can give the number which goes on as 2.12345… as an example."

Of the participants, S5 and S7 used the statement "*numbers that are not fractions*" to define irrational numbers. S5 attempted to explain this response via an example. The dialog between the researcher and this student was as follows:

> "………
> Researcher: You said they are numbers that cannot be written in the form of fractions, but I could not get what you meant by this statement.
> S5: For example, 29/18 is a number that can be written in the form of a fraction, but we cannot express irrational numbers in that form."

When one said irrational number, S7 thought of the number π and added that *they are numbers that cannot be written as a/b"*. The dialog between the researcher and this student was as follows:

> Researcher: What do you think is an irrational number?
> S7: When one says irrational number, I think of the number π. We cannot write these numbers as a/b. For example, the number π goes on as 3.12349123…
> Researcher: You said the number π is an irrational number, how did you determine that it is so?
> S7: I cannot write it as a/b. I determined that it is an irrational number because I cannot put it in the form of a fraction.

Justification of S7 was as given above. S8 and S9 defined irrational numbers as "*numbers with digits which do not form a periodic pattern after the decimal point.*" In the light of the example, S8 stated, "*0.123576… is irrational. That is, non-periodic numbers are irrational.*" While S9 said, "*these numbers continue to infinity in a non-periodic pattern.*" S2 and S6 described irrational numbers saying "*numbers with decimal parts that continue to infinity.*" Below is an excerpt from the interview with S6:

> Researcher: What do you think when you hear the word irrational numbers? How would you define those numbers?
> S6: I can define irrational numbers as the little slice of numbers among the rational numbers.
> Researcher: What do you mean? I couldn't exactly understand what you said. Would you provide an alternative explanation?
> S6: I think we can describe them as numbers that cannot easily be placed on the number line and that continue to infinity.
> Researcher: I see. Can you then give an example to these numbers?
> S6: The number π can be an example. And the number 6.18765449….
> Researcher: Okay, how did you determine that these are irrational numbers?
> S6: I mean, because they continue to infinity.
> Researcher: Is there any other reason for this?
> S6: No, there isn't.

Explanation of S6 was as given above. As to S10, this student acknowledged not having sufficient knowledge on irrational numbers, yet thought of repeating numbers when one said irrational numbers and added, "*I don't know what irrational number means. I cannot describe them. I cannot give an example, either. But, I suppose irrational numbers are repeating numbers.*"

Student responses were analyzed collectively, and it was concluded that students defined irrational numbers usually as "*numbers that are not rational*" and as "*numbers that we cannot write in the form of a fraction.*" Apart from these, it was seen that students also used such statements as "*numbers that continue to infinity, repeating numbers*", which refer to some certain properties of irrational numbers although they do not fully define irrational numbers.

"How do students explain the difference between rational numbers and irrational numbers?"

An analysis of the student responses regarding the difference between rational and irrational numbers yielded table 3.

Table 3. Student responses regarding the difference between rational and irrational numbers

Codes	Students
Rational numbers can be written in the form of a fraction but irrational numbers cannot be written.(C)	S2, S3, S5, S7
A given number is rational if its digits after the decimal point form a periodic pattern, but it is irrational if those digits are not periodic. (IC)	S1, S9
If a given number can move from inside the radical to outside the radical, it is rational; if it cannot move, then it is irrational. (IC)	S8
Decimal parts of rational numbers do not continue to infinity, but those of irrational numbers do. (I)	S4
Rational numbers can be shown on the number line, but irrational numbers cannot be shown. (I)	S6
I don't know. (NE)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC), No explanation: (NE)

A collective analysis of the student responses demonstrated that according to four students (S2, S3, S5, S7), the main difference of rational numbers from irrational numbers was their eligibility for being written in the form of a fraction. An excerpt from the interview with the students is given below:

> Researcher: Do you think there is a difference between rational numbers and irrational numbers?
> S3: Hmm. Yes, I think so.
> Researcher: What kind of a difference is it?
> S3: We can write rational numbers in the form of a fraction, but it is not possible when it comes to irrational numbers.
> Researcher: What do you mean, can you explain it further?
> S3: We can write rational numbers as a/b, but we cannot write irrational numbers in this form.

Some of the students (S1, S9) said that rational and irrational numbers can be told apart depending on whether the digits after the decimal point follow a periodic pattern i.e. regular pattern or not. These students' statements were coded as IC because they did not emphasize the number of digits after the decimal part. Another student (S8) determined whether a given number was rational or not depending on whether it was possible to move it from inside the radical to outside the radical:

> Researcher: Is there a difference between rational and irrational numbers?
> S8: Yes, there is.
> Researcher: What kind of a difference is it? What do you think?
> S8: For example, the number √16 is rational because it can move outside the radical as 4. However, √5 is irrational because it is not a square that can be simplified.

This students explanation was as coded as IC because he or she did not be aware of square root of a number is always equal to a reel number and can be express with a decimal number. According to one of the students (S4), being infinite or not would be considered as the criterion for determining the type of the number. The interview with the subject student was as follows:

> Researcher: Is there a difference between rational and irrational numbers?
> S4: Yes, there is. Rational numbers do not continue to infinity, but irrational numbers do. .
> Researcher: I see. For example, is the number 2.010101… irrational?
> S4: Yes.

One student (S6) had a totally different approach and determined the type of a given number depending on its place on the number line. This student said that rational numbers could be demonstrated on the number line, but irrational numbers could not. An excerpt from this interview is given below:

> Researcher: Is there a difference between rational and irrational numbers?
> S6: Yes, there is.
> Researcher: How did you determine that?
> S6: Based on whether they can be shown on the number line or not.
> Researcher: Hmm. Can you explain it further? I couldn't exactly get what you said.
> S6: Rational numbers can be shown on the number line, but irrational numbers cannot be shown.

One student (S10) acknowledged having no idea. An analysis of the student responses showed that some students gave correct responses while more than half of the students gave either incorrect or incomplete responses. One student, on the other hand, did not make any explanations. Two students described the difference based on whether the digits to the right of the decimal point were regular or not while another student said the difference stemmed from whether given numbers moved outside the radical or not. One of the students whose response was incorrect evaluated numbers depending on whether those numbers continued to infinity. Another student with an incorrect response suggested that irrational numbers could not be shown on the number line.

"How Do Students Classify Numbers as Irrational or Rational?"

Students were asked to classify the numbers 0.16, $13.\overline{2}$, -2.010101…, 14.01010304405…., √9, 2√3 and 22/7 as well as the number π as irrational or not and to justify their decision. For analyzing the students' responses, the definition given in course books, namely "*numbers that cannot be written as the ratio of two integers are defined as irrational numbers*" was taken into consideration. Student responses that included this statement were marked as correct. As to the responses which explained certain properties of irrational numbers (non-repeating decimals, digits after the decimal point do not form a periodic/regular pattern, decimal part goes on forever…), yet did not provide the expected definition were marked as incomplete. Student responses that were not accurate, on the other hand, were marked as incorrect. The resultant findings are given in the table below. In the light of the evaluation of the student responses about the number 0.16, correctness, incorrectness or incompleteness of the statements can be presented as in table 4.

Table 4. Students' classification of the number 0.16 as rational or irrational and their justification for this classification

Codes	**Students**
It is a rational number because it can be written in the form of a/b as a rational number/fraction. (C)	S1, S2, S3, S4, S5,S6, S7, S8
It is a rational number because there is no irregularity and it is not infinite. (IC)	S9
It is a rational number because it is not a repeating number. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

All the participant students defined the number 0.16 as a rational number, which is accurate; however, they justified their responses in different ways. S1, S2, S3, S4, S6 and S7 justified their answers saying "*because it can be written in the form of a/b as a rational number*" while S5 and S8 stated "*because it can be written in the form of a fraction*" as their reason. As to S9 and S10, they said that it is not an irrational number and explained the reason for it as follows: "*It is not an irrational number because there is no irregularity, and it is not infinite" (S9), "It is not an irrational number because I think it is not a repeating number*" (S10).

Student responses regarding the number $13.\overline{2}$ are given in table 5.

Table 5. Students' classification of the number $13.\overline{2}$ as rational or irrational and their justification for this classification

Codes	**Students**
It can be expressed as a rational number. (C)	S1, S3, S4, S5, S6, S7
It is a rational number because it has a periodic decimal expansion. (IC)	S8
It is a rational number because it is finite. (I)	S9
It is not a rational number because it is infinite. (I)	S2
It is not a rational number because it is a repeating number. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

If student responses are examined, it is seen that six students responded correctly, three students responded incorrectly while one gave an incomplete response. S1, S3, S4, S5, S6 and S7 indicated that the number $13.\overline{2}$ is not irrational, since they could write it as a rational number. Students explained the possibility of writing a number as a rational number in different words. For example, S4 used a formula during the process of converting the number into a rational number, and it is shown in the dialog below:

> Researcher: Do you classify the number $13.\overline{2}$ as irrational or not?
> S4: I think it is a rational number.
> Researcher: How did you determine that it is a rational number?
> S4: The number continues to infinity, it repeats as 13.222222…. I can write this number in the form of a rational number.
> Researcher: How do you write it?
> S4: First of all, I subtract the digits to the left of the decimal point from the whole number. Secondly, I write as many 9's as the repeating digits as the denominator. (132-13)/9= 119/9. In this way, I can write it as a rational number.

In another response, S8 explained the criterion for identifying numbers as rational or not saying "*A number is not an irrational number if it has a periodic decimal expansion.*" The interview with the subject student is given below:

> Researcher: Do you classify the number $13.\overline{2}$ as irrational or not? How would you justify your answer?
> S8: It is not an irrational number because it has a periodic decimal expansion.
> Researcher: What do you mean by periodic decimal expansion, can you explain it further?

S8: It continues to infinity in a regular pattern.

Although S9 identified the number as rational, this student justified this response saying *"the fact that the number is finite indicates that it is rational." "Being infinite"* was suggested as the criterion when defining a rational number in the response of S2 while according to S10, "being a repeating number" was the criterion met by irrational numbers. An analysis of the student responses showed that students who provided correct explanations used a formula (The whole number – Non-repeating part / As many 9's as the repeating digits to the right of the decimal point and as many 0's as the non-repeating digits after the decimal point) to determine whether a number was rational or not. Furthermore, it is possible to say that they checked whether the repeating decimal part of a number continued to infinity in a periodic way to determine its rationality. Some of the students with correct responses to the question justified their response incorrectly by saying *"it is rational because the repeating decimal is finite."* Another interesting finding demonstrated that according to one student, any infinite number was irrational regardless of whether the part to the right of its decimal point was repeating or not. Student responses regarding the number -2.0101… were as presented in table 6.

Table 6. Students' Classification of the number -2.0101… as rational or irrational and their justification for this classification

Codes	Students
It is a rational number because the part after the decimal point is periodic/repeating. (IC)	S1, S3, S5, S6, S7, S8, S9, S10
It is an irrational number because the part after the decimal point continues to infinity. (I)	S4
It is an irrational number because it cannot be written in the form of rational number. (I)	S2

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

When student responses were examined, it was seen that all the students except for two (S2, S4) defined the given number as not irrational either because "*the part after the decimal point is repeating*" or because "t*he part after the decimal point is periodic*." These students' statements were coded as IC because they emphasized only periodical part after the decimal point. S2 defined the number as irrational saying "*it cannot be written in the form of rational number,*" and S4 named it irrational and suggested *"it continues to infinity"* . An analysis of the student responses regarding the number 14.01010304405… yielded table 7.

Table 7. Students' classification of the number 14.01010304405… as rational or irrational and their justification for this classification

Codes	Students
It is not a rational number because the part to the right of the decimal point continues to infinity, and it cannot be written in the form of a rational number. (C)	S2, S4, S6, S7, S9
It is not a rational number because the part right to the decimal point continues with a non-periodic pattern. (IC)	S1, S5
It is not a rational number because the repeating part to the right of the decimal point is not given in the form of a regular expression. (I)	S3, S8
It is a rational number because it is a complex number that is not repeating. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

Nine of the participant students were able to identify 14.01010304405… as an irrational number. However, when they were asked to justify their responses, there were some incomplete or incorrect explanations in addition to the correct ones. S2, S4, S6, S7 and S9 considered the subject number irrational because the part to the right of the decimal point continues to infinity, and it cannot be written in the form of a rational number. Two students' (S1 and S5) statements were coded as IC because they stated only *"rational number because the part right to the decimal point continues with a non-periodic pattern".* They did not emphasize the number of digits after decimal point that was not countable. Three of the students (S1, S5, S9) indicated "*it is not a rational number because the part right to the decimal point continues irregularly with a non-periodic pattern.*" Although S3 and S8 could name the number as irrational their explanation for their responses was inaccurate. They made an inaccurate explanation about irrational numbers by saying "the repeating part to the right of the decimal point is not given in the form of a regular expression." S10, on the other hand, gave an incorrect response. An analysis of the student responses revealed that students tended to use such statements as "*continuing to infinity, not being written in the form a rational number and digits to the right of the decimal* *point following a non-periodic pattern*" when they tried to identify an irrational number. An analysis of the student responses regarding the number √9 yielded table 8.

Table 8. Students' classification of the number √9 as rational or irrational and their justification for this classification

Codes	Students
It is a rational number because 9 is simplified as 3. (C)	S1, S2, S3, S4, S5, S6, S7, S8, S9, S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

Student responses regarding the question about the number √9 were justified through correct explanations. An analysis of some excerpts from student responses revealed that students suggested the following explanations: "*It is not an irrational number because it is simplified as 3, and 3 is a rational number.*" (S7) "*It is not an irrational number because if we simplify √9, we get 3, and 3 is written in the form of a rational number.*" (S3).
An analysis of the student responses regarding the number 2√3 yielded table 9.

Table 9. Students' classification of the number 2√3 as rational or irrational and their justification for this classification

Codes	Students
It is not a rational number because 3 cannot move outside the radical sign. (IC)	S2, S3, S6, S7
It is not a rational number because √3 cannot move outside the radical sign as an integer, and its decimal part is not periodic. (IC)	S1, S5
It is a rational number because it can move outside the radical. (I)	S8, S9
It is a rational number because it can be converted to a rational number. (I)	S4
It is a rational number, but I don't know why. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

Given the student responses, it is clear that although six students (S1, S2, S3, S5, S6, S7) said it is not a rational number, they had difficulty explaining the reason for their response. Four of the students (S4, S8, S9, S10), on the other hand, gave incorrect responses. Two of the students named √3 as irrational adding that "*it is irrational because it moves outside the radical sign as an integer, and its decimal part is not periodic.*" The students whose expressions were coded IC, had some difficulties in some statements. They were not aware of the square root of a number was a reel number and could be represented as decimal number. The excerpt below exemplifies this approach:

> Researcher: Do you think 2√3 is a rational or irrational number?
> S6: It is an irrational number.
> Researcher: Why do you think it is an irrational number?
> S6: It is because √3 cannot move outside the radical.

S8 and S9 classified the subject number as a rational number. When they were asked to explain why they did so, they put the whole term under the radical sign and said it was a rational number. The responses regarding √9 and 2√3 evidence that students believed if it was not a perfect square, a number could not be completely extracted out of the radical sign, and hence it could not be a rational number. The responses also demonstrate that students lacked the knowledge that it is not a rational number because it cannot be written as a ratio of two integers.
An analysis of the student responses regarding the number 22/7 yielded table 10.

Table 10. Students' classification of the number 22/7 as rational or irrational and their justification for this classification

Codes	Students
It is a rational number because it is written in the form of a numerator and denominator. (IC)	S1, S2, S3, S4, S5, S6, S7
It is not a rational number because it has a non-periodic pattern. (I)	S8, S9
It is not a rational number because we take the number π as 22/7, and π is an irrational number. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

It can be said that the majority of the student responses were correct, and they were able to justify their responses through correct explanations. Seven of the students justified their responses saying "*It is a rational number because it is written in the form of a numerator and denominator.*" These statements were coded as IC because they did not define numerator and denominator as integers. Incorrect student responses unveiled two significant points. Students S8 and S9 performed division with the number 22/7, and after finding the first three digits to the right of the decimal point, they believed the number would continue to infinity in a non-periodic pattern and hence, classified the subject number as irrational.

> Researcher: What would be your comment regarding the number 22/7?
> S9: Just a moment. I need to perform an operation. Hmm. If we divide 22 by 7, it goes on as 3.142… in an irregular pattern.
> Researcher: So what do you think? Is it a rational number or an irrational number?
> S9: It continues to infinity in a non-periodic pattern, therefore it is an irrational number.

Another important point is that one student thought the number pi was equal to 22/7. This was discussed in the dialog given below:

> Researcher: What would be your comment regarding the number 22/7?
> S10: 22/7 equals to the number pi and pi is an irrational number. Accordingly, 22/7 is an irrational number.
> Researcher: How did you determine that the number pi equals to 22/7?
> S10: This was what we were told in the class. We took the number pi as 22/7 for an operation.

An analysis of the student responses regarding the subject question revealed that students had the misperception that 22/7 equaled to the number π and that they classified 22/7 as irrational because they made calculation errors when dividing 22 by 7. An analysis of the student responses regarding the number π yielded table 11.

Table 11. Students' classification of the number π as rational or irrational and their justification for this classification

Codes	**Students**
If π is equal to 3.14 assumed, then the continuation of 3.14 will be irregular and will go to infinity (C)	S7
It is not a rational number because its decimal part continues to infinity in an irregular pattern. (C)	S1, S2, S3, S5, S8, S9
It is not a rational number because it continues to infinity. (IC)	S4, S6
It is not a rational number because it is a repeating number. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

Although all the students named the number π as irrational, some of them used incomplete or incorrect statements to explain their responses. S7's explanation was correct. An excerpt from the interview with this student is given below:

> Researcher: What would you say about the number π?
> S7: It is an irrational number.
> Researcher: Why?
> S7: Because, although we usually assume it to be approximately 3.14, the number 3.14 does not terminate, and it has a non-periodic pattern. As a result, it cannot be written in the form of a fraction or a rational number.

In addition S1, S2, S3, S5, S8, S9 were able to explain that it is an irrational number because its decimal part continues to infinity in an irregular pattern. Other student responses revealed that continuance to infinity was regarded as a criterion for classifying a number as irrational or not. An excerpt from the interview with S4, which confirms this finding, is provided below:

> Researcher: What type of a number is the number π, rational or irrational?
> S4: It is an irrational number.
> Researcher: Why?
> S4: Because pi continues to infinity.
> Researcher: Hmm, I see. Do you think it is essential to continue to infinity for a number to be considered irrational?
> S4: Yes.

Although S10 gave a correct response, this student's explanation for the subject response was incorrect. This student stated that the number is irrational because "*it is a repeating number*." It is clear from the student responses that students sometimes made mistakes when they were asked to classify a number as rational or irrational and to explain the reason for their responses. Most of the students checked *"whether the part to the right of the decimal point continues to infinity"* to determine the type of the number. Although the majority of the responses were correct, explanations for the responses were either incomplete or incorrect.

"How Do Students Explain the Relationships between Sets of Numbers?"

This question required students to explain the relationships between sets of numbers. As a consequence, four of the students provided correct responses. When the student responses were examined, the responses shown in Figure 1, 2, 3 and 4 were obtained. When the responses of S1, S4, S5 and S7 were examined, it was seen that they named the real numbers as the largest set of numbers and within this large set of numbers, they defined rational and irrational numbers as two separate subsets.

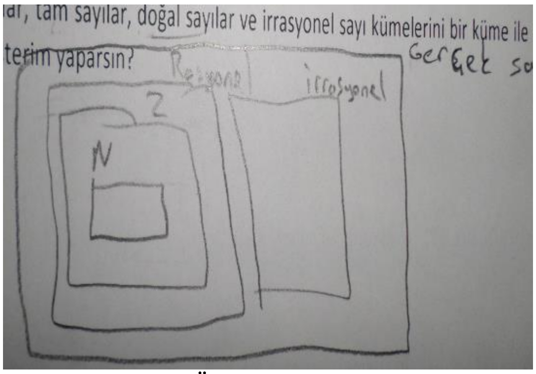

Figure 1. Ö1's correct answer

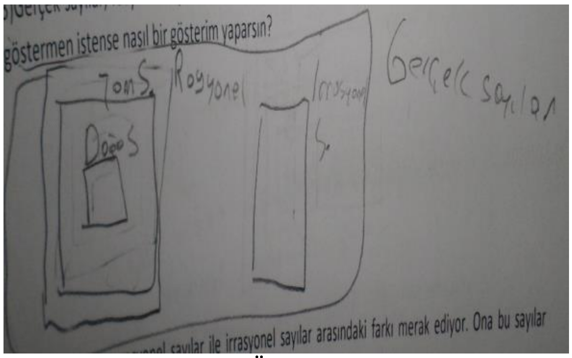

Figure 2. Ö4's correct answer

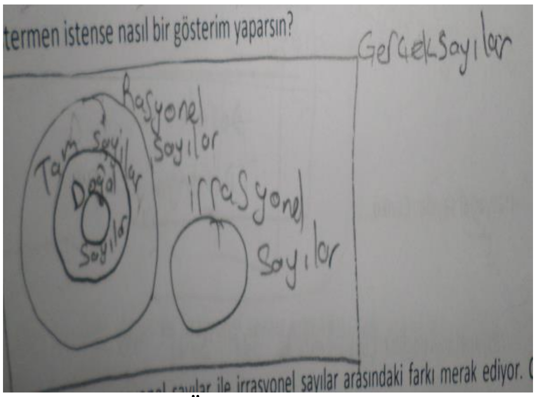

Figure 3. Ö5's correct answer

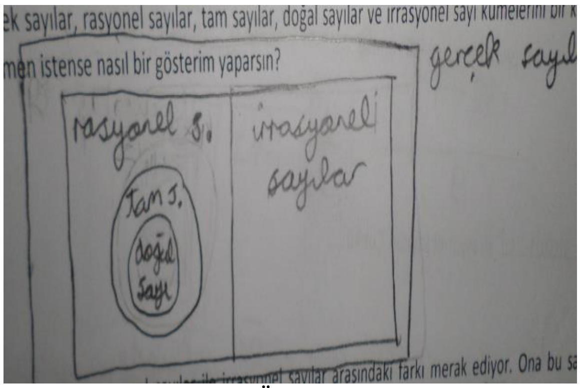

Figure 4. Ö7's correct answer

It can be deduced from the interviews conducted to investigate how the students reflected on the issue that some of the students who gave correct responses approached the sets of numbers from a holistic perspective. Students indicated that real numbers involve all the sets of numbers, and then, they divided numbers into two categories as rational and irrational. They regarded integers and natural numbers as part of the set of rational numbers. One student tried to provide representation on the basis of the smallest set of numbers. At first place, this student defined natural numbers as the smallest set of numbers. Following this, the student drew the set of integers and the set of rational numbers. Initially, this student demonstrated the set of irrational numbers separately, but then decided that all the aforementioned sets of numbers should be handled under the set of real of numbers.

An analysis of the other student responses showed that students had some incomplete or incorrect ideas as to the relationships between sets of numbers. Samples from those responses are provided in Figure 5 and 6. S2 and S8 understood the relationship between rational and irrational numbers correctly; however, they misunderstood the relationship between the systems of rational numbers, integers and natural numbers.

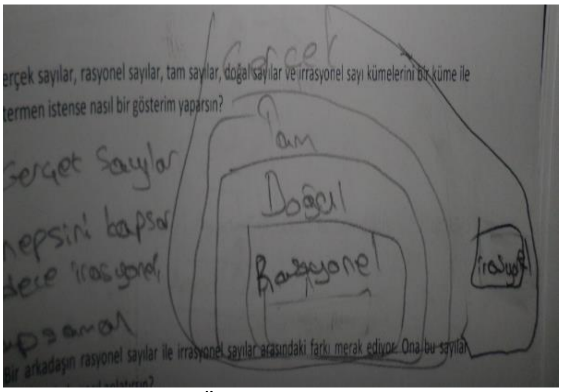

Figure 5. Ö2's incorrect answer

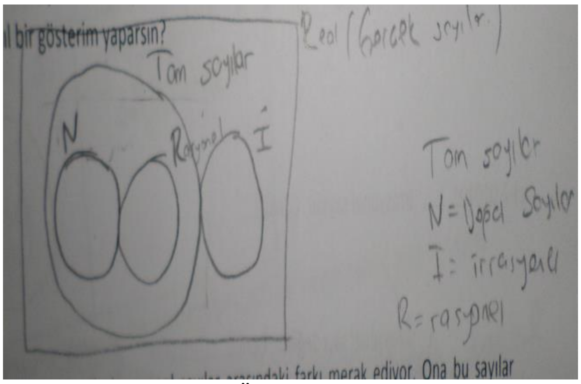

Figure 6. Ö8's incorrect answer

S8 perceived the set of natural numbers and rational numbers as independent sets. An excerpt from the interview is given below:

> Researcher: In your opinion, what is the relationship between the given sets of numbers?
> S8: Real numbers involve all sets of numbers. And integers involve natural numbers and rational numbers.
> Researcher: Hmm, how did you determine that?
> S8: I guess the set of integers is a larger group. Therefore, it involves both natural numbers and rational numbers, but I suppose irrational numbers form an independent set of numbers.

S2 provided a wrong representation for the sets of numbers, since this student could not see the relationship between the set of natural numbers and of rational numbers. This was discussed in the dialog given below:

> Researcher: In your opinion, what is the relationship between the given sets of numbers?
> S2: Real numbers involve all sets of numbers. As for irrational numbers, they are part of the set of real numbers, yet are differentiated from other sets of numbers. I mean, the other sets of numbers should be handled separately.
> Researcher: How would you represent the other sets of numbers you mentioned?
> S2: OK. Integers are a larger group than natural numbers and rational numbers. Therefore, the largest set is integers. They are followed by natural numbers and rational numbers, but I'm not sure which of these two sets is larger. I guess natural numbers form a larger set.

In figure 7 and 8, S3 and S6 put irrational numbers in a separate part. Those students regarded irrational numbers as a totally independent set of numbers. S6 could connect all the sets of numbers correctly except for irrational numbers while S3 made a range of mistakes when trying to demonstrate the relationship between the sets of numbers.

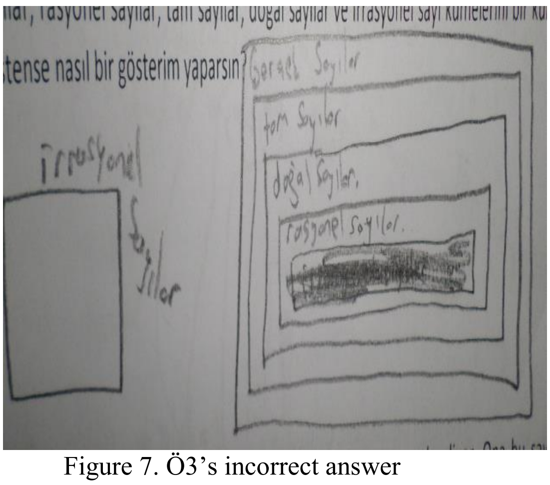

Figure 7. Ö3's incorrect answer

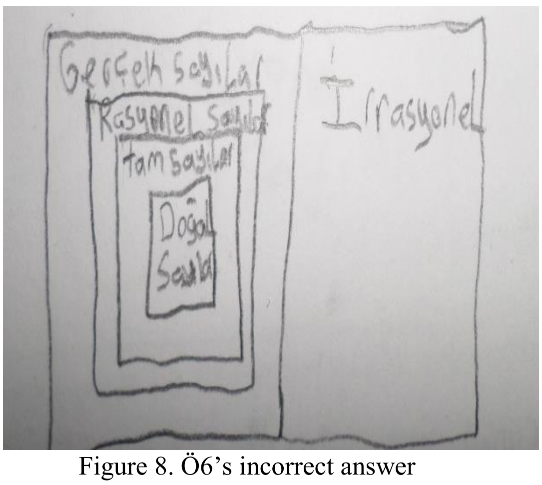

Figure 8. Ö6's incorrect answer

In the relevant interviews with the students, S6 justified the subject response as follows:

> Researcher: What is the relationship between the given sets of numbers?
> S6: I suppose real numbers involve all numbers except for the set of irrational numbers. Natural numbers appear as the subset of integers while the set of integers is actually a subset of rational numbers.
> Researcher: How did you determine that?
> S6: Let's study an example. For example, 1, 2 etc. can be natural numbers, but integers can also be negative. Therefore, they form a larger set. As to rational numbers, they include numbers written in the form of fractions.
> Researcher: Then, where are the irrational numbers?
> S6: Irrational numbers are totally independent; they do not fit in any of these sets.
> Researcher: How did you determine that?
> S6: For example, the number pi. It does not belong to any of these sets of numbers; therefore we should handle it separately.

Other students namely S9 and S10 demonstrated the numbers as given in figure 9 and 10.

Figure 9. Ö9's incorrect answer

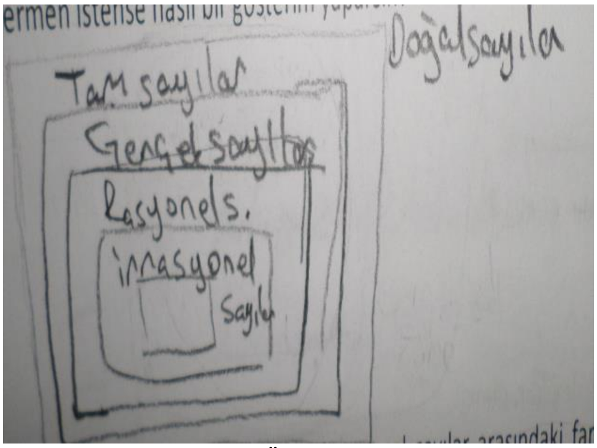

Figure 10. Ö10's incorrect answer

If student responses are analyzed, it is possible to say that S9 and S10 could not see the relationship between the number systems. S9 considered rational and irrational numbers as a whole and believed that natural numbers involved both of those sets of numbers. S9 also suggested that each of these three sets of numbers was actually a subset of integers and indicated that real numbers involve all of these sets of numbers. S10, on the other hand, stated that irrational numbers were a subset of all sets of numbers and demonstrated them without mentioning any relationship among other sets of numbers.

"How Do Students Determine Whether the Results of Operations with Rational and Irrational Numbers Are Irrational or Rational?

In this section, student opinions regarding some expressions that involved operations with rational and irrational numbers were identified. They were asked to evaluate some statements, mark them as true or false, and explain the reason for their response. These statements were: The sum of any two irrational numbers will always be an irrational number, the sum of any two rational numbers will always be a rational number, multiplying one rational number by another must produce a rational result, and multiplying one irrational number by another must produce an irrational result.

If a given student evaluated a statement correctly and was able to justify it correctly, her/his answer was marked as correct. However, if the response was correct but the explanation was wrong, it was marked as incomplete or incorrect according to the explanation. Finally, if both the response and the explanation were wrong, the statement was marked as incorrect. The resultant findings are presented under the relevant headings as given below.

Findings regarding the Statement "The Sum of Any Two Irrational Numbers Will Always Be an Irrational Number"

The participant students were expected to evaluate the statement, "the sum of any two irrational numbers will always be an irrational number." An analysis of their responses yielded Table 12.

Table 12. Student responses regarding the statement "the sum of any two irrational numbers will always be an irrational number

Codes	Students
If I add two irrational numbers, the sum can be a rational number. (C)	S10
If I add two irrational numbers, the sum can be a rational number, $\sqrt{5}+\sqrt{4}=\sqrt{9}$. And $\sqrt{9}$ moves outside the radical as 3. (I)	S8
If I add two numbers with decimal parts following a non-periodic pattern, the sum will always be an irrational number. (I)	S3, S4, S6, S9
If I add two radicals, the sum will also be radical.(I)	S1, S2, S5
If we add two numbers which cannot be written in the form of a fraction, we only make these numbers greater in value. (I)	S7

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

When student responses were examined, it was seen that only S10 responded correctly. However, this student was not able to justify his or her response and failed explain the reason. This was discussed in the dialog given below:

> Researcher: If we add two irrational numbers, is the sum always an irrational number?
> S10: It's not true.
> Researcher: Why?
> S10: I think it is not always so. Yes, there must be some irrational examples to such operations, but there might be operations with rational results. This is why it is false.

S8 gave the correct response, but justified this response incorrectly. Thus the statement coded incorrectly because S8's explanation was based on an incorrect way of thinking. S8 had problems in adding the irrational numbers. An excerpt from the interview with S8 is given below:

> Researcher: If we add two irrational numbers, is the sum always an irrational number?
> S8: I don't think it always produces this result.
> Researcher: Why do you think so?
> S8: Hmm, for example $\sqrt{5}+\sqrt{4}=\sqrt{9}$. And $\sqrt{9}$ moves outside the radical as 3.
> Researcher: OK. Thank you.

Although other students made some explanations, they could not find the correct response. Four of the students (S3, S4, S6, S9) gave an example of numbers with decimal parts following a non-periodic pattern and suggested that the sum would have a decimal part following a non-periodic pattern if those numbers were added.

> Researcher: If we add two irrational numbers, is the sum always an irrational number?
> S3: Right.
> Researcher: Why?
> S3: If, for example, I add two irrational numbers such as 3.1421822…+2.81125311…, the sum will always be an irrational number.

The other three students (S1, S2, S5), on the other hand, tried to justify their responses through examples on radical numbers.

> Researcher: Will the sum of any two irrational numbers always be an irrational number?
> S1: I think this is a true statement because $2\sqrt{3}+2\sqrt{3}=4\sqrt{3}$. $2\sqrt{3}$ and it is an irrational number.
> Researcher: Hmm, I see. Is there a counterexample?
> S1: I think this operation produces an irrational number. This is always the case.

An excerpt from the interview with S7 whose response was incorrect is given below:

> Researcher: If we add two irrational numbers, is the sum always an irrational number?

> S7: I think this is a true statement.
> Researcher: How did you understand that it is true?
> S7: Because if we add two numbers which cannot be written in the form of a fraction, we only make these numbers greater in value.

If student responses are analyzed, it can be said that they lacked sufficient knowledge regarding irrational numbers because, as it is known, the sum of any two irrational numbers can be a rational number (e.g. $(1+\sqrt{3})+(-\sqrt{3})=1$) Furthermore, students who provided examples can be said to lack sufficient knowledge on operations with radical numbers.

Findings regarding the Statement "The Sum of Any Two Rational Numbers Will Always Be a Rational Number"

An analysis of the student responses regarding the statement, "the sum of any two rational numbers will always be a rational number" yielded table 13.

Table 13. Student responses regarding the statement "the sum of any two rational numbers will always be a rational number

Codes	**Students**
If I add two rational numbers, the sum will be a rational number. (C)	S1, S3, S5, S6, S9
If we add two numbers that can be written as a/b and c/d, we only make these numbers greater in value, and their type remains the same. (C)	S7
If I add two rational numbers, the sum will be a rational number, for example $\sqrt{4}+\sqrt{16}=\sqrt{20}=2\sqrt{5}$ (I)	S8
Rational numbers do not continue to infinity. They produce rational numbers when added up. $2\sqrt{3}+2\sqrt{3}=2\sqrt{6}$ (I)	S4
When added up, they are not written as irrational numbers, they remain the same in all cases. $\sqrt{5}+\sqrt{5}=\sqrt{10}$ (I)	S2
The result may not always be a rational number. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

Six of the students gave correct responses. Although the majority preferred to explain their responses through examples, one student chose to make a verbal explanation. An excerpt from the interviews with one of the students who responded correctly is presented below:

> Researcher: Will the sum always be a rational number if we add two rational numbers?
> S1: Yes, it is rational.
> Researcher: Why?
> S1: Because 2/3 + 1/6 = 5/6. As both are rational, it is certain that the result will be a rational number.
> The interview with S7 who chose to provide a verbal explanation is given below:
> Researcher: Will the sum always be a rational number if we add two rational numbers?
> S7: Yes, it is rational.
> Researcher: Why?
> S7: If we add two numbers that can be written as a/b and c/d, we only make these numbers greater in value, and their type remains the same.

As to S8, this student was found to have some conceptual problems regarding the conception and the four basic operations with radicals:

> Researcher: Will the sum always be a rational number if we add two rational numbers?
> S8: Yes, it is rational.
> Researcher: How did you determine that it is so?
> S8: I concluded so because, for example $\sqrt{4}+\sqrt{16}=\sqrt{20}=2\sqrt{5}$.
> Researcher: Is $\sqrt{20}$ a rational number?
> S8: Yes.
> Researcher: How did you determine that it is so?
> S8: Because I added two rational numbers.

S4 was also found to have a similar incorrect statement. The other two students (S2, S10) gave incorrect responses. An excerpt from the interview with S2 is presented below:

Researcher: If we add two rational numbers, will the sum always be a rational number?
S2: I think it is a wrong statement.
Researcher: Why??
S2: Because when added up, they are not written as irrational numbers, they remain the same in all cases.
Researcher: Can you give an example?
S2: $\sqrt{5} + \sqrt{5} = \sqrt{10}$

When student responses are analyzed, it is clear that six students knew the correct response and could justify it. Two of the students, on the other hand, could not explain their responses although they gave the correct response. Finally, the other two students gave incorrect responses. As was the case in previous questions, the students had some difficulties in radical numbers.

Findings regarding the Statement "Multiplying One Irrational Number by Another Must Produce an Irrational Result"

When student responses were analyzed, it was seen that four of them could explain their responses via correct explanations. The resultant findings are presented in table 14.

Table 14. Student responses regarding the statement "multiplying one irrational number by another must produce an irrational"

Codes	Students
If we multiply two radical numbers, the result will be a rational number. (C)	S1, S2, S4, S5, S7, S8
The result may not always be an irrational number. (IC)	S10
Multiplication of two numbers with non-periodic patterns produces a result with a non-periodic pattern. (I)	S3, S6
No explanation. (NE)	S9

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC), No explanation: (NE)

It is seen that six students' responses were correct, and these students justified their responses through examples. In their examples, students preferred to use radical numbers. An excerpt from the interview with S7 is presented below:

Researcher: Will the result always be an irrational number if we multiply two irrational numbers?
S7: It is a wrong statement. It can also be a rational number.
Researcher: Why?
S7: Because $\sqrt{3}.\ \sqrt{3} = \sqrt{9} = 3$ is a natural number. $\sqrt{2}.\ \sqrt{8} = 4$

Although S10 responded correctly, this student could not provide any explanations. S3 and S6 indicated that multiplying two irrational numbers with a non-periodic pattern after the decimal point must produce an irrational result. On the basis of this argument, they concluded that the statement was true. The interview with S3 is given below:

Researcher: Will the result always be an irrational number if we multiply two irrational numbers?
S3: Yes, it will.
Researcher: How did you understand that?
S3: Because when multiplied, they will yield a result with a non-periodic pattern, that is, an irrational number.
If we multiply 3.4567915…. by 6.78910210…, the result will again be an irrational number.

S9 did not provide any explanations.

Findings regarding the Statement "Multiplying One Rational Number by Another Must Produce a Rational Result"

When student responses were analyzed, it was seen that nine of them gave correct responses. Only of one them made an incorrect explanation about the response. The resultant findings are presented in Table 15.

Table 15. Student responses regarding the statement "multiplying one rational number by another must produce a rational"

Codes	**Students**
If we multiply two rational numbers or decimal expressions by one another, the result will always be a rational number. (C)	S1, S2, S3, S4, S5, S6, S7, S8, S9
The result can sometimes be rational and sometimes irrational. (I)	S10

Correct statement: (C), Incorrect statement: (I), Incomplete statement: (IC)

An analysis of the student responses revealed that nine students were able to give correct responses; however, their explanations varied. Seven of them preferred to use examples when explaining their responses. They chose to use rational numbers and decimal expressions in these examples. An excerpt from the interview with S1 whose explanation was based on examples is given below:

> Researcher: Will the result always be a rational number if we multiply two rational numbers?
> S1: Yes, it will be rational.
> Researcher: How did you understand that?
> S1: Because if we multiply rational numbers, the result will always be a rational number because we do not change the number itself. For example, ¾. 2/3= ½.

S7 preferred to use a verbal explanation to justify the subject response:

> Researcher: Will the result always be a rational number if we multiply two rational numbers?
> S7: Yes, it will be rational.
> Researcher: How did you understand that?
> S7: Because I don't deem it likely that the result would be an irrational number if two numbers that can be written in the form of a fraction are multiplied. For example, if I assume that rational numbers are positive and irrational numbers are negative, the result won't be negative when two positive numbers are multiplied.

S8, on the other hand, preferred to provide an explanation by means of square root:

> Researcher: Will the result always be a rational number if we multiply two rational numbers?
> S8: Yes, that's right.
> Researcher: How do you know that? Can you give an example?
> S8: $\sqrt{16}.\sqrt{4}= \sqrt{64}=8$

S10 who gave an incorrect response emphasized the word "always", but could not give any examples as given:

> Researcher: Will the result always be a rational number if we multiply two rational numbers?
> S10: The result can sometimes be rational and sometimes irrational. Therefore, it is wrong to say that this is "always" the case.
> Researcher: How do you know that? Can you give an example?
> S10: I don't know any examples.

If student responses are analyzed in general, it is clear that they all gave correct responses and tended to use examples to justify their responses.

Results, Discussion and Conclusion

The present study explored students' skills and knowledge on the definition of irrational numbers, classification of a given set of numbers as irrational or not, the relationships between sets of numbers and irrational numbers, and operations with irrational numbers. The first research question investigated how students defined irrational numbers. It was seen that half of the student responses were correct while the other half were either incorrect or incomplete. When correct student definitions were examined, it was revealed that the most common expressions used in the definitions was "*numbers that cannot be written in the form of a fraction*" or "*numbers that are not rational*". Students' definitions of irrational numbers also involved incorrect and incomplete expressions such as "*decimal part continues to infinity, a repeating number*". For example, it was found out during the interviews that one of the students who made an incomplete explanation saying "*numbers that continue to infinity*" had some misconceptions which caused this student to say "*numbers that continue to infinity and that cannot be easily shown on the number line.*" It was determined from the interviews the statement students used *"numbers that continue to infinity"* was meant that *"the number of digits is countless*". If they did not express this completely their expressions were coded as incomplete. It is possible to say that students tended to make explanations through examples rather than providing a definition of irrational numbers. Another interesting finding is that one of the students defined repeating numbers as irrational numbers. Teachers' tendency to teach rational numbers by means of examples, particularly via those about repeating numbers, instead of discussing the definition with their students might have caused students to respond in this way. Putting more emphasis on practicing the subject with relevant examples without thinking about the meaning of the concept first appears as another possible reason for student responses. Participants were eighth grade students during the study term, and they were going to take national entrance exam for high school at the end of the semester, therefore they placed more emphasis on finding the correct answer than the concept itself. As a consequence, this might have influenced their responses. In teaching the rational number, it is important at this point to show the emphasis rational numbers can be expressed through repeating or non-repeating decimal expansions and to illustrate them by various examples in the light of the definition. It is essential that in the representation of decimal numbers, the right to the decimal point continues to infinity in a repeating pattern, and it is expressed as a rational number. In a similar fashion, the difficulties students experienced regarding the definition of irrational numbers might be due to the fact that they were instructed the subject only through a limited number of examples ($\sqrt{2}$, $\sqrt{3}$, π). A literature review shows that both students and pre- and in-service teachers experience some difficulties when defining irrational numbers (Arbour, 2012; Arcavi, et al., 1987; Baştürk & Dönmez, 2008; Ercire, 2014; Ercire, et al., 2016; Fischbein, et al., 1995; Güven, et al., 2011; Güler, 2017; Kara & Delice, 2012; Sirotic, 2004). In addition to the difficulties they experienced when defining these numbers, the participants of the past studies were understood to suggest definitions which were based on representations and intuitions and which were far from being formal. It was clarified that their definitions of irrational numbers mostly entailed such statements as "*numbers that are not rational*" or "*numbers that have infinite and non-periodic decimal parts.*" In this regard, the findings of the present study correspond to those in the relevant literature.

The second research question required students to explain the difference between rational numbers and irrational numbers. Findings for the first research question were reverberated in the second question. In their explanations about the difference between rational numbers and irrational numbers, students used some incorrect knowledge which they previously displayed when defining the term and which entailed such misunderstandings as "*if the decimal part of a number continues to infinity, it must be an irrational number; rational numbers decimal part does not continue the infinity and the decimal part of irrational numbers continue to infinity.*" Four of the students indicated that the difference between the subject numbers depended on "*whether they can be written in the form of a fraction*" while the other students experienced a range of difficulties when they were asked to make an explanation. Another noteworthy finding was that they thought irrational numbers could not be shown on the number line. In a study conducted by Kanbolat (2010) with high school students, pre- and in-service teachers, the participants stated that "*a decimal part with infinite digits*" was the difference between irrational and rational numbers. In this regard, the study findings correspond to those in the literature.

The third research question asked students to classify numbers as rational or irrational. Students had difficulty during this task, and some inconsistencies were detected in their knowledge on the subject. Some believed that repeating decimal numbers expressed as rational numbers had countable digits and oppositely all decimal numbers that had not countable digits were irrational numbers. An interesting finding was that they considered 22/7 an irrational number and also equal to the number π. The students might have responded that way because they might have generalized having a decimal part with a non-periodic pattern, a characteristic of irrational numbers, to repeating numbers. Apart from this, they might have given incorrect responses, since both in the instruction process and reference books the number pi is usually taken as 3 or 3.14 for the sake of convenience. It was also found out that students found it hard to justify their answers although they could define radical numbers as irrational. The respondent students were confused, since radical numbers cannot be simplified in the form of the ratio of two integers. Therefore, they interpreted this fact as if the numbers inside the radical could never be extracted out of the radical sign. This perception might be a consequence of the instruction process. If the literature is reviewed, the existing studies are also seen to emphasize similar consequences (Adıgüzel, 2013; Arbour, 2012; Çiftçi, et al., 2015; Ercire, 2014; Fischbein, et al., 1995; Güven, et al., 2011; Kara & Delice, 2012; Peled & Hershkovitz, 1999; Sirotic, 2004; Temel & Eroğlu, 2014).

Another research question aimed to explore how students explained the relationship between the sets of numbers. Four students responded correctly to this question. As to the rest of the students, they could see the relationship between some sets of numbers although they could not understand the relationship between other sets of numbers. Additionally, there were also some students who could not understand the relationship between sets of numbers. Some students handled the set of irrational numbers as an independent set and could not see their connection with real numbers. Besides, some students were of the opinion that irrational numbers were not real numbers. Another finding revealed that although students could understand the relationship between rational, irrational and real numbers, they misperceived the relationship between other sets of numbers. A literature review shows that similar problems have been identified in other study populations as well (Adıgüzel, 2013; Arbour, 2012; Ercire, et al., 2016; Kara & Delice, 2012). In their study, Yeşildere and Türnüklü (2007) asked students questions about the visual representation of the relationship between sets of numbers and found out that 70% of the participant students gave incorrect responses. 91% of the students whose responses were incorrect stated rational numbers involved irrational numbers. Similarly, in the present study, there were some students who gave the afore-mentioned response. Given this information, it can be said that students cannot make sense of the relationship between rational numbers and irrational numbers. In the past studies, respondents defined irrational numbers as "a wide range of rational numbers which proves that those respondents believed irrational numbers involved rational numbers (Kanbolat, 2010). In this regard, findings of the current study are consistent with the relevant literature. The lack of proper learning environments in which relationships among sets of numbers are clearly explained might have produced such student responses. Alternatively, instruction of the sets of numbers independently of each other might have paved the way for the student responses in the study.

The last research question investigated how students determined whether the results of operations with rational and irrational numbers were irrational or rational. When answering this question, students found it hard to provide reasons for which the given statement should be considered true or false. In the relevant literature both teachers and pre- and in-service teachers were found to have difficulty assessing such statements as true or false (Adıgüzel, 2013; Arbour, 2012; Ercire, 2014; Güler, 2017; Güler, et al, 2012, Güven, et al., 2011; Sirotic & Zazkis, 2007a). This might be due to the fact that they did not take into consideration the fact that irrational numbers are actually real numbers. These results can also be explained by students' lack of experience in interpreting statements of this type.

Recommendations

On the basis of the study findings, we can make some recommendations. They are as follows: The concept of number is among the key components of mathematics curricula. The concept of irrational number might, by its nature, be difficult. Nonetheless, rational numbers within the system of real numbers should be addressed more intensely in the instruction process in order to ensure a better understanding of irrational numbers (Sirotic & Zazkis, 2007b). Therefore, before the concept of irrational number is introduced in the eighth grade, students might be engaged in activities on the set of rational numbers and on other sets of numbers which would help them build a better understanding. For students, eighth grade can be regarded as the transition process to formal mathematics. According to Piaget, in this period, students transit from the stage of concrete operations into the stage of abstract operations. Moving to irrational numbers after strengthening students' knowledge on other sets of numbers, which they have predominantly studied up to the eighth grade, is important as this helps those students enrich their knowledge. It can be recommended that this process be enriched with activities and problems that would attract the interest of students while also motivating them. Teachers can encourage the use of calculators in order to overcome the misconceptions that radical numbers which cannot be simplified or written in the form of a real number. This can be used to help students understand different sets of numbers and see relationships between them. In this process, they should identify the numbers that remain in the radical as real numbers and examine whether it is rational or irrational depending on its value. In the instruction of sets of numbers, providing interesting and short examples from the history of mathematics and giving examples about the challenges scholars experienced in the past when they were working towards identifying rational and irrational numbers have been found to increase students' motivation and help students develop a positive attitude (Furinghetti, 1997; Jardine, 1997). In this regard, such practices will certainly make positive contributions to the instruction process.

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