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Abstract

This study aims to investigate the obstacles in eighth-grade students' understanding of integer exponents using a mixed method research design. A total of 165 eighth-grade students were given a paper-pencil task and clinical interviews were conducted with 12 students. The findings indicated that achievement of the participants was low, especially in the zero and negative exponents. Students made errors that generally originated from the definition of exponentiation as repeated multiplication with natural numbers and under-developed conceptions of additive and multiplicative structures. As a result, the students overgeneralized the rules that are true for positive integer exponents to the other exponent expressions. Another crucial result was that most of the students did not know the meaning of zero exponents due to various obstacles due to the identity element of addition or the absorbing element of multiplication. Furthermore, students made various errors when undertaking operations with exponent expressions due to the confusion with additive and multiplicative structures in the operations of exponential expressions.

Introduction

In the extension of a number system, "new" numbers generally contain rules and definitions that differ from previously recognized numbers (e.g., Levenson, 2012). For example, in the first years of school, students are most familiar with natural numbers. However, the addition of zero expands the number system from natural numbers to whole numbers, and requires students to modify or update the previous definitions and images in their mind, but they cannot always achieve this. For example, Levenson, Tsamir, and Tirosh (2007) reported that sixth-grade students have difficulty in deciding whether zero is an even number. In this sense, the process of extending the number system is not an easy issue for both teachers and students.

Although most students perceive exponents as a new number set, in fact, exponents enable them to abbreviate repeated multiplications of the same number. The concept of exponents is difficult to acquire since it requires considering the relationship between symbols, meanings, and the algorithmic properties of exponentiation (Pitta-Pantazi, Christou & Zachariades, 2007). In this process, procedural knowledge is not sufficient to undertake the appropriate calculations in order to find the value of exponential expressions without understanding the logic behind algorithms and the hierarchy between number systems. For instance, when computing the numerical value of an exponential expression, students often multiply the base with the exponent to arrive at a correct solution (e.g., Cengiz, 2006). Yet, exponentiation includes certain rules related to base and power. This situation leads to problems resulting in the students becoming confused and failing to remember these rules. In furtherance of this idea, studies have revealed that students perceive exponents as complicated and difficult concepts, and they lack connection with everyday life (İymen & Duatepe-Paksu, 2015; Şenay, 2002). However, exponentiation is important for the understanding of advanced mathematical concepts, such as exponential functions, logarithm, and calculus. Furthermore, exponentiation is crucial to interpret some of the models in different fields, such as population control, radioactive decay, problems of inflation, musical scales, algebra, and complex analysis (Ellis, Ozgur, Kulow, Dogan &Amidon, 2016; Weber, 2002).

The related literature contains many studies on advanced mathematical topics, such as functions, limits, and infinity; however, there are relatively few studies on prospective teachers' understanding of exponents (Kontorovich, 2016; Levenson, 2012; Zazkis & Kontorovich, 2016) and students' understanding of exponentiation. Some researchers examined secondary school students' mental constructions of exponents (Mullet & Cheminat, 1995; Munoz-Sastre & Mullet, 1998; Pitta-Pantazi et al., 2007) and college students' understanding of exponents (Cangelosi, Madrid, Cooper, Olson & Hartter, 2013), but there is insufficient research into middle school students' understanding of exponents (Avcu, 2010). The research focusing directly on exponents as an autonomous mathematical object is very limited and so is the investigation of students' understanding of the negative sign, particularly in the context of exponential notation (Cangelosi et al., 2013; Kieran, 2007). Instead, more studies are conducted regarding exponents in terms of examining learners' conceptions of functions and logarithms (Confrey & Smith, 1995; Weber, 2002). Yet, students in middle school start to learn exponents and use exponential notation in various contexts [Common Core State Standards Initiative (CCSSI), 2010; Ministry of National Education (MoNE), 2013; National Council of Teachers of Mathematics (NCTM), 2000]. In this respect, it is significant that a strong foundation is constructed to support middle school students' understanding of exponents in order to prevent obstacles to their understanding of exponentiation in the following years.

Obstacles result from the persistent errors that learners make irrespective of their age, country, culture, and level of success in math (Brousseau, 2002). Obstacles have critical importance in learning since they force learners to make changes and adapt in some aspects of their thinking to overcome the contradiction. For this reason, the identification and characterization of obstacles in learners' understandings is crucial to the analysis and organization of didactical situations for the learners (Brousseau, 2002). From this point of view, in this study, eighth-grade students' understanding of integer exponents was examined by determining the obstacles that hinder students' conceptions of exponents. The research questions that guided this study were: (i) What is the achievement level of eighth-grade students in integer exponents regarding zero exponents, negative exponents, and operations with exponential expressions? and (ii) What are the obstacles hindering eighth-grade students' understanding about integer exponents and how can these obstacles be categorized?

Literature Review

Obstacles in Mathematics Education

Errors and failures are not always the effect of ignorance, uncertainty or chance (Brousseau, 2002; Modestou & Gagatsis, 2007); rather, they can result from the misapplication of a piece of previous knowledge which was interesting and correct in some contexts, but which in another context becomes incorrect (Brousseau, 2002). Such kinds of errors are not erratic and unexpected, but are reproducible and persistent. Errors of this kind can be evaluated as the indicator of an obstacle. From this perspective, the notion of obstacle was introduced by the French philosopher and scientist Bachelard (1938), who considered that obstacles are the heart of cognition for historical improvement of scientific thought and individual learning.

Duroux (1982) presented a list of necessary conditions to describe the meaning of the term obstacle: (i) An obstacle is a piece of knowledge or a conception rather than a difficulty or lack of knowledge. (ii) This piece of knowledge produces information which can be appropriate in a particular context. (iii) However, these responses become incorrect outside of the specific context. (iv) This piece of knowledge is grounded on both occasional contradictions and the establishment of a better piece of knowledge. (v) After its inaccuracy has been recognized, it continues to crop up in an untimely, persistent way (Brousseau, 2002). In this sense, Duroux distinguishes between difficulty and obstacle. On the other hand, Brousseau (2002) proposed that an obstacle apparent in learners' erroneous responses may not be due to chance or ignorance. In addition, these errors can result from "a way of knowing, a characteristic conception, coherent if not correct, an ancient knowing that has been successful throughout an action domain" (Brousseau, 2002, p. 84). Similarly, Mallet (2013) defines a cognitive obstacle as "a situation where an existing mental structure is appropriate for one domain but causes difficulty with learning in another domain due to incompatibility with the new situation or concepts" (p. 152). In brief, an obstacle is a knowledge which is useful in some contexts, but when applied in a new context or problem situation, it is inadequate or leads to cognitive conflict between two contexts (Brousseau, 2002; Herscovics, 1989).

As an example of an obstacle, children sometimes propose *multiplication makes bigger* by overgeneralizing their experiences in natural numbers to all numbers (Gagatsis & Kyriakides, 2000; Graeber & Campbell, 1993). Similarly, "*numbers must be measures of something*" is an obstacle and in this argument, students cannot consider negative numbers (Brousseau, 2002). Furthermore, the idea of "*a product should be larger than its factors*" becomes an obstacle when a student is faced with a problem of the form $4.\blacksquare = 3$. In a comprehensive study, Bishop, Lamp, Philipp, Whitacre, Schappelle, and Lewis (2014) identified elementary school children's cognitive obstacles about integers, such as (i) *negative rejected*, (ii) *subtrahend < minuend*, and (iii) *addition cannot make smaller*; *subtraction cannot make larger*. In the literature, there are also studies that identify the forms of obstacles as *epistemological*, *cognitive*, and *didactical nature*. Obstacles in epistemological nature are internal to mathematics itself (Brousseau, 2002; Sierpinska, 1987), those of a cognitive nature come from learners' conceptualization and abstraction processes (Cornu, 1991; Dubinsky, 1991; Sfard, 1991; Tall & Vinner, 1981), and finally, obstacles of a didactic nature originate in the nature of teaching and learning processes (Brousseau, 2002). Although researchers have classified different types of obstacles, the term cognitive obstacle was used in the current study to mean understanding or knowledge that once supported a learner's thinking about exponents. As a result, in the current study, the source of an obstacle was not distinguished with specific terminology as adopted by other researchers.

In this study, due to its more applicable and objective nature for the data, Duroux's list of five necessary conditions was used in the determination of the obstacles operationally. However, when applying this list operationally, some conditions were evaluated by adopting a different perspective. The aim of this study was not to encourage students to recognize their mistakes. For this reason, it was decided not to use condition (v) "after its inaccuracy has been recognized, it continues to crop up in an untimely, persistent way". Another important point was related to condition (ii), which refers to the knowledge provided by learners must be correct in some context. In this study, some students' conceptions about the meaning of exponentiation is shaped via some statements, such as *An exponentiation is multiplication of numbers*. Using this statement, the students calculated 2^5 as 2.5 *or* 5.2. This symbolic representation is not applicable in any context, but in spoken form, this statement brings multiplication in exponential expressions with natural numbers to the minds of the students. For this reason, when determining the obstacles, some memorized rules in students' explanations were also evaluated even if they seemed incorrect in any context.

Existing Studies on the Understanding of Exponents

Exponent is an important topic in middle school mathematics since it provides the learners with a background to understanding of more complex ideas. In recent years, in secondary and college level, there has been an increasing interest in educational research in the conceptions of advanced mathematical concepts, such as exponential functions and logarithm (e.g., Cangelosi et al., 2013; Ellis et al., 2016; Kontorovich, 2016; Weber, 2002). Although topic of exponents is central to many college mathematics courses, such as calculus, differential equations and complex analysis, there has been comparatively little research that focused on middle school students' learning and understanding of exponents (Avcu, 2010; Ellis et al., 2016, Ellis, Ozgur, Kulow, Williams, & Amidon, 2015). Weber (2002) found that university students have difficulties concerning the rule of exponentiation and connecting it to rules of logarithms, and other studies reported that teachers have problems in understanding and teaching exponents (Levenson, 2012; Zazkis & Kontorovich, 2016).

Exponentiation is defined as repeated multiplication with natural numbers (Goldin & Herscovics, 1991; Weber, 2002). Similarly, exponents are generally presented in the textbooks using the repeated multiplication approach and students already perceive exponents from that point of view (Ellis et al., 2015; Ellis et al., 2016). In this approach, students need to examine the relationship between variables to perform repeated multiplication and connect the operation to the notation of exponentiation. Exponent expressions involving positive whole number bases and exponents are first introduced in elementary school. In the following years of middle school, the definition of exponentiation is expanded to include negative integers. This situation requires students to rethink the intuitive definition of exponentiation as repeated multiplication in order to understand the abstract mathematical definition of zero and negativity. If they can understand the extended definition appropriately, they can use the rules of operation in the extended domain (Levenson, 2012).

Mathematics educators and mathematicians agree that the use of examples in teaching and learning as a communication tool between learners and teachers is very useful in helping students comprehend mathematical concepts (e.g., Watson & Mason, 2005). Some researchers suggest that certain examples of each mathematical concept are more commonly used in learning and teaching (Hershkowitz, 1990; Schwarz & Hershkowitz, 1999). These examples are called prototypes; however, due to the students' overexposure to prototypes, their concept images often only involve prototype examples. As a result, students may tend to identify a concept by referencing the critical features of one or a few prototypical examples (Pitta-Pantazi et al., 2007). Hence, students focus on the properties of prototypical examples, rather than referencing the formal definitions of mathematical concepts (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). In this regard, Pitta-Pantazi et al. (2007) categorized 202 secondary school students' understanding of exponents into three levels. At Level 1, the students used prototypes as repeated multiplication. At Level 2, they extended the prototype to include positive and negative rational exponents. In the meanwhile, they can understand $\mathrm{a}^{\mathrm{b}}\mathrm{a}^{\mathrm{c}} = \mathrm{a}^{\mathrm{b+c}}$ only when a, b and c are positive integers. However, at Level 3, they expanded the prototype over all rational exponents (positive or negative). In conclusion, the results of previous studies indicate that students commonly experience confusion by overgeneralizing the rules and definitions of positive integer exponents to the exponents including negative exponents or zero exponents (Cangelosi et al., 2013; Pitta-Pantazi et al., 2007) due to the influence of the prototypical examples. In such situations, students could not construct mental match between mental images of previously learned exponents and their extended forms. Thus, they inevitably make errors that generally originate from the lack of exponential number sense and overgeneralizing the rules that are true for natural numbers, integers and rational numbers to exponents and roots (Duatepe-Paksu, 2008). Similarly, some students misinterpret the identity element of addition, zero. As a result, they consider that the 0^{th} power of an expression is equal to the expression itself (Cengiz, 2006; Crider, 1998). In another important study, Cangelosi et al. (2013) investigated the persistent errors when simplifying exponential expressions of 904 freshman and sophomore undergraduate students. They grouped persistent errors both qualitatively and quantitatively, and their work indicated that students make persistent errors due to negative signs, spoken language, grouping, and notation.

In contrast, some researchers have concluded that teachers present inappropriate generalization about the rules of multiplication and power properties of exponents due to the strong understanding of exponential growth as repeated multiplication (Davis, 2009; Presmeg & Nenduardu, 2005). Not only the teachers but also the students have difficulties when generalizing to negative integer exponents by adopting the repeated multiplication perspective (Davis, 2009). Considering this limitation, Ellis et al. (2016) proposed the covariation approach to exponential growth as an alternative to the repeated multiplication approach. To this purpose, the authors constructed an Exponential Growth Learning Trajectory to trace learners' initial and developing understanding of exponential growth. Based on Thompson's (2008) study, they defined exponential growth as the notion that the rate at which the function changes with respect to x is proportional to the value of the function at x. From this perspective, exponential growth includes two quantities that covary continuously. In brief, the formation of exponents requires well-structured procedural and conceptual knowledge since past research has stressed secondary school students' or college students' errors or conceptions about exponents.

Method

A survey research design (Fraenkel & Wallen, 2006) was used in line with the research questions of the study. Additionally, some of the participants were selected to be interviewed with the aim of seeking an answer to the second research question; therefore, mixed methods research was performed to address the research questions of the study.

Context and Participants

In some studies conducted in the light of the PISA (Programme for International Student Assessment) results, researchers noted that the socio-economic and socio-cultural factors are the basic factors that determine the quality of education in Turkey (e.g., Aydın, Sarıer, & Uysal, 2012; Yıldırım, 2012). Furthermore, it has been reported that a large number of the Turkish students with poor performance in PISA come from low socio-economic background (Büyüköztürk, Çakan, Tan, & Atar, 2014; Yılmaz-Fındık & Kavak, 2013; Yolsal, 2016). From this perspective, selecting students from low socio-economic status backgrounds attending public middle schools presents an opportunity to identify the obstacles that block learners' correct conceptions.

The sample of the study comprised 165 eighth-grade students aged 13 to 14 attending two middle schools. The reason for the grade selection was related to the nature of the objectives identified in the Middle School Mathematics curriculum (MoNE, 2018) because there is a heavy emphasis on objectives about exponents in the eighth grade. According to the curriculum, the fifth graders are expected to show the square and the cube of a natural number as exponential and finding value (e.g. $3^2 = 3.3 = 9$), sixth graders to express repeated multiplication of a natural numbers by itself in exponential quantities and to determine their value (e.g.$2^6 = 2.2.2.2.2.2 = 64$), seventh graders to express the multiplication of integers with themselves in exponential quantities (e.g.-$2^4 = -(2.2.2.2) = -16$), and finally, eighth graders to calculate both negative and positive exponential expressions (e.g., $3^{-2} = \frac{1}{3}.\frac{1}{3} = \frac{1}{9}$), to understand the basic rules of exponential expressions, and to create equivalent expressions [e.g., $a^n.a^m = a^{n+m}$; $\frac{1}{a^n} = a^{-n}$; $a^n = \frac{1}{a^{-n}}$; $\frac{a^n}{a^m} = a^{n-m}$; $(a^n)^m = a^{n.m}$; $a^0 = 1$; $(a.b)^k = a^k.b^k$; $(\frac{a}{b})^k = \frac{a^k}{b^k}$, $(b \neq 0)$]. In conclusion, based on the objectives in the related curriculum, an eighth grader should be able to perform all calculations in exponent expressions and understand the algorithmic rules of exponents at the end of the term.

Data Collection Procedure

Brousseau (2002) suggested that teachers and researchers should select suitable tasks that would make learners utilize a specific piece of knowledge to reveal a learning obstacle. In the current study, ten questions were prepared based on the results of previous studies on exponents (e.g., Cangelosi et al., 2013; Crider, 1998; Levenson, 2012; Pitta-Pantazi et al., 2007) to identify students' achievement and learning obstacles in regard to integer exponents. Some researchers have tended to focus on the comparison of exponential expressions (Avcu, 2010; İymen & Duaptepe-Paksu, 2015; Pitta-Pantazi et al., 2007) while others conducted learners' understanding of zero exponents (Kontorovich, 2016; Levenson, 2012) and the negative exponent (Zazkis & Kontorovich, 2016). Based on the results of these studies and objectives identified in the mathematics curriculum, questions were prepared on the presence of (i) zero exponent (e.g., 3^0), (ii) negativity in bases (e.g., $-2^5 \; or \; (-2)^5$), and (iii) negative exponents (e.g., $2^{-5} \; or \; 2^5$). Expert opinions were elicited from two academicians in the Mathematics Education Department of a public university in Turkey. They were asked to evaluate the questions in terms of consistency with the national objectives, appropriateness for the grade levels, and clarity. Subsequently, the task was revised taking the experts' comments into consideration, and the preliminary version was piloted with a class of eighth graders. It is noteworthy that feedback elicited from the students during the task administration, and their answers to the questions provided considerable insight into the basic issues, and played a crucial role in finalizing the task before the actual data collection stage. The related details about the questions are presented in Table 1.

Fischbein (1993) claimed that mathematical concepts have three interacting components of formal, algorithmic, and intuitive. These aspects were taken into consideration during the preparation of the questions of exponents in the task. The first question (Q1) aimed to gain insight into students' knowledge about zero exponents, as well as the meaning of the minus sign which precedes the base. The second question (Q2) intended to understand how the students interpret the meaning of an exponent number with a negative exponent. Differently, the third question (Q3) concerning exponential expressions that have the same base as the expression in Q2 was posed to the students with the purpose of observing how students interpret the negative exponent in two similar expressions. The remaining questions referred to operations with integer exponents and were posed to understand how the students made calculations with exponents. For example, the value of addition of exponential expressions with same bases and exponents was asked in Q4, and the value of addition of exponential expressions with same bases but different exponents was asked in Q5. During the administration of the task, the students were not posed questions in relation to the subtraction of exponential expressions in consideration of the complexity of the coexistence of a minus sign in subtraction and the minus sign of the base or exponent. Three more questions (Q6-Q7-Q8) were directed to the multiplication of exponential expression based on the nature of numbers in the bases and exponents. Similarly, two additional questions (Q9-Q10) were posed concerning the division of exponential expressions. In the multiplication and division of exponential expressions questions, pairs of numbers with common divisors were generally selected for the exponents or the bases (e.g., 5-5, 9-3), which provided the opportunity to interpret how students overgeneralized the algorithms of division with the numbers with common divisors.

Table 1. Types of questions in the exponent task

No	Questions	Types
Q1	$-3^0 =?$	Negativity and zero exponent
Q2	$2^{-4} =?$	The negative exponent
Q3	$2^4 =?$	The positive exponent
Q4	$3^5 + 3^5 + 3^5 =?$	Addition of exponential expressions with the same bases and exponents
Q5	$2^4 + 2^{-2} + 2^3 =?$	Addition of exponential expressions with the same bases
Q6	$5^{-2}.5^4 =?$	Multiplication of exponential expressions with the same bases
Q7	$5^{-4}.6^4 =?$	Multiplication of exponential expressions with different bases and exponents
Q8	$3^{-4}.4^{-4} =?$	Multiplication of exponential expressions with the same exponents
Q9	$5^6/\,5^{-3} =?$	Division of exponential expressions with the same bases
Q10	$9^3/3^3 =?$	Division of exponential expressions with the same exponents

In the task, for the following two reasons, the questions were prepared to include integer bases and powers; first, *exponentiation* is a difficult mathematical topic since it requires considering the relationship between notations, algorithms, meanings, and properties of exponents (Pitta-Pantazi et al., 2007) and second, a conceptual understanding of exponentiation requires an in-depth knowledge about other number sets (e.g., natural numbers, integers, and rational numbers). Accordingly, the related literature indicates that different uses of negatives in exponential expression cause contradictions in students' initial interpretations of exponents (e.g., Cangelosi et al, 2013). The current study aimed to determine the serious obstacles that prevent students from correctly responding to the questions of integer exponents instead of focusing on how students establish relationship among all number sets. Specifically, this study investigated how students interpret base and exponent in an exponent expression and undertakes operations with them.

At the beginning of the data collection, the participants responded to the questions in the task in this process, the students were encouraged to provide written expressions about how they found the values for each question. Motivated by their teacher offering them extra points in their final mathematics examination, almost all students actually provided written explanations for their answers. The students were informed that even if they thought their answers were wrong, they should give an explanation for their answers. Since the study had a qualitative nature, clinical interviews were implemented to allow a deeper investigation and help the researchers to "enter the learners' mind (p. 430)" (Zazkis & Hazzan, 1999). Hence, after the analysis of the participants' written responses to the questions, interviews were held with 12 eighth-grade students for the purposes of clarifying and amplifying their reasoning and gaining a better understanding of the obstacles that triggered their incorrect responses. From the students who had made many incorrect responses to the questions, the reproducible and persistent errors were examined and 12 students were selected as shown in Table 2.

In the interviews, follow-up questions were posed to learn more about the students' thinking. For example, different questions about exponents were asked to obtain detailed information about their understanding (e.g., Could you explain the differences, if any, between 2^{-6}, 2^6 and -2^6?). The interviews were of a semi-structured nature; thus, the same questions were not posed to every student. In the interview process, the students were first asked to think aloud and comment on their answers. Covering many aspects of the exponent concept, the interviews were conducted a week after the administration of the test, and lasted approximately 25-30 minutes. The interview questions were prepared in association with the follow-up questions of Hunting (1997), such as "Can you solve this question again?", "Can you tell me what you are thinking?", "How did you find this value?", "What is the meaning of this notation?", and "Which question do you find more difficult and why?" While answering the questions, the participants were allowed to examine their own written responses whenever they wanted. However, in order to avoid biased thoughts/opinions, no feedback was given concerning the correctness of the students' written responses during the interviews.

Table 2. The selection of the interviewees

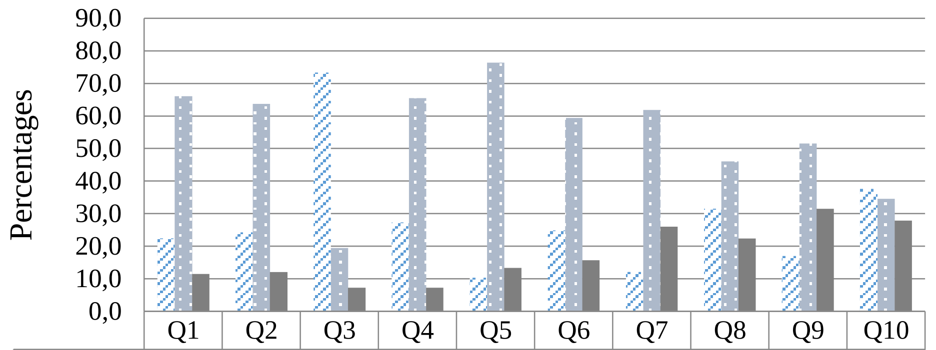
Categories	The number of students	Example responses
The meaning of exponentiation	4	Two interviewees were selected from the students who did not use the definition of exponentiation as repeated multiplication with natural numbers in their written responses. Instead, these students only multiplied the base and the exponent in the expression (e.g., 2^6 means $2.6 = 12$)
		Another two interviewees were selected from the students who incorrectly used the definition of exponentiation as repeated multiplication with natural numbers in all questions. Normally, the exponent of a number shows how many times to use the number in a multiplication. However, these students interpreted 2^6 as 6.6, instead of 2.2.2.2.2.2.
The use of negativity in exponential expressions	8	Three interviewees were selected from the students who proposed that if there is a positive and a negative in an exponentiation, the result becomes negative in all questions including negativity. These students interpreted 2^{-6} as $-2^6 = -(2.2.2.2.2.2)$.
		Three interviewees were selected from the students who rejected negativity in the case in which there is an even number in the exponent. These students attributed same meaning the negatives in the base and the exponent in an exponential expression (e.g., 2^{-6} means $2^6 = 2.2.2.2.2.2$)
		Two interviewees were selected from the students who inappropriately interpreted the negativity in the exponent as a signal to form a reciprocal somewhere within the expression (e.g. 2^{-6} as $\pm 2^{\frac{1}{6}}$).

Data Analysis

Based on the purpose of the study, the students' written responses to the questions in the task were analyzed, and the interviews were transcribed. First, their answers for each question were evaluated as correct, incorrect or blank to calculate corresponding frequencies and percentages (*Note*: If a particular mistake was made in all questions by the students, their percentages were calculated in order to determine the possible obstacles in students' understandings). When calculating the ratio of students' errors in any question, importance was given to whether three or more students who were not in same class provided written explanations for their similar responses. Thus, an attempt was made to eliminate students' random responses that were not justified when calculating minimum acceptable percentages. Then, all written responses in the tasks and verbal explanations in the interviews were qualitatively examined to detect obstacles underlying participants' incorrect responses, and consequently certain categories were produced (Dey, 1993). When developing the categories, some resources including inferences made on the basis of the data, research questions, theoretical considerations, and previous studies were taken into account (Dey, 1993). In this sense, the students' written work was read to establish a relationship between the literature and data before producing the codes. Thus, the students' written explanations were a large part of the data supporting the results of the study. Furthermore, the results of the previous studies on the exponentiation, integers, decimals, and fractions were utilized in the process of category creating. In addition, the interview transcripts were examined in order to support and strengthen the findings. The interview data was analyzed by open coding of the videotapes (Strauss & Corbin, 1998). The focus of this coding was based on students' ways of understanding concerning exponentiation. Thus, the obstacles in students' understandings of integer exponents were grouped as (i) the meaning of exponentiation, (ii) zero exponent, (iii) negative exponents, and (iv) operations with integer exponent expressions. In order to ensure the validity and reliability of the study, the codes were discussed by the researchers and the data were separately coded. Finally, the codes were compared and discussed until a total consensus was reached.

Findings

This section presents a general picture of the students' achievement on exponents based on their scores in the task. Each correct response received 1 point, while blank and incorrect responses received no points. According to the scores, only 4 students (3.5%) correctly responded to all the questions in the task. More than 65% of the students received less than 3 points; thus, a large percentage of students were unable to answer the three questions in the task. Based on these percentages, it is argued that the students had difficulty integer exponents. More details about the students' achievement and their correct, incorrect and blank responses to each question are depicted in Figure 1.



	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Correct	22,4	24,2	73,3	27,3	10,3	24,8	12,1	31,5	17,0	37,6
Incorrect	66,1	63,6	19,4	65,5	76,4	59,4	61,8	46,1	51,5	34,5
Blank	11,5	12,1	7,3	7,3	13,3	15,8	26,1	22,4	31,5	27,9

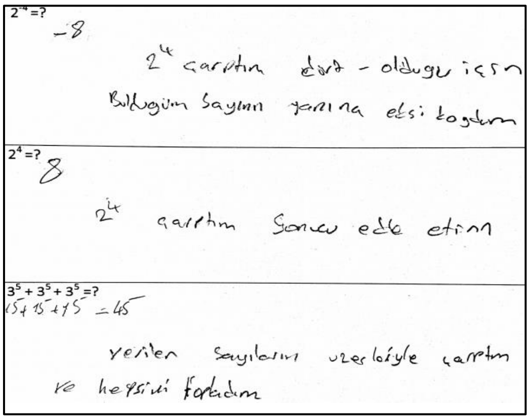
Figure 1. The percentages of correct, incorrect and blank responses from the students

The students' achievement levels concerning the questions about exponents varied from 10.3% to 37.6% (excluding the percentages of Q3). It was also seen that they commonly provided incorrect responses to Q1 about the zero exponent. This situation could be evaluated as an indicator of the difficulties the students face when working with zero exponents. On the other hand, the difference between the percentages of correct responses to Q2 (24.2%) and Q3 (73%) revealed the role of negativity in exponent on the students' responses. Another important finding of the study is that most of the students incorrectly responded to or left blank the questions Q4 to Q10 that involved operations with integer exponents. In brief, Figure 1 provided general information about what kind of questions the students correctly and incorrectly responded; however, it did not allow a detailed analysis of how students reached a value for an exponent or why they made incorrect calculations. Thus, their written explanations in the task, and interview data were qualitatively analyzed. Based on the analysis, considering the findings the obstacles in students' understanding of exponents were grouped into four main categories related to (i) the meaning of exponentiation, (b) zero exponents, (iii) the negativity in exponents, and (iv) operations with exponents.

Obstacles regarding the Meaning of Exponentiation

An Exponentiation is Multiplication of Numbers

An exponential notation consists of two parts; the base and exponent. This notational form is a powerful way to express the repeated multiplication of the same number. For instance, 2^4 means 2.2.2.2. However, in the task, seven students (4%) multiplied the base and the exponent in all questions (see Figure 2) because they proposed that an exponentiation means multiplication. Exponentiation has a multiplication meaning; however, they interpreted multiplication in a way that was not repeating. As a result, they only multiplied the base and the exponent (e.g., $2.4 = 8$ for 2^4). In written responses, some of them wrote following explanations: "Exponentiation requires multiplication operation. So, I multiplied the numbers." In this respect, it can be concluded that these students were not aware of the functional structure of the exponentiation. As a result, they considered the base and the exponent in an exponent expression as two separate numbers. Furthermore, they multiplied the exponent and the base to obtain the value in all questions. Figure 2 shows how some students calculated the value as $3.5 + 3.5 + 3.5 = 45$ when responding to $3^5 + 3^5 + 3^5$ in Q4.



Translation of written explanations
I multiplied 2 by 4. Since there was a negative sign in front of 4, I added it to the result I found.

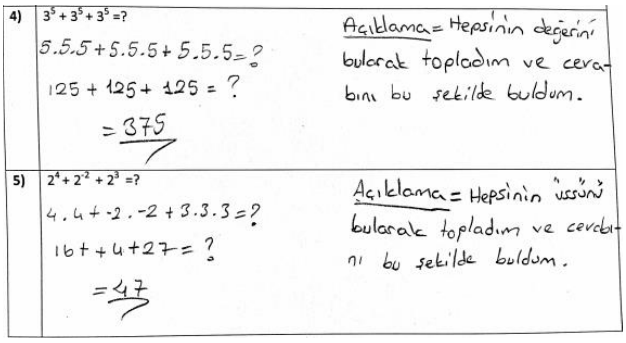
I multiplied 2 by 4 and found the result.

I multiplied the bases with the exponents. Then, I added them all.

Figure 2. Example responses from the students to Q2, Q3 and Q4

An Exponentiation is a Repeated Multiplication

In the questions, 11 students misapplied the definition of exponentiation as repeated multiplication with natural numbers since they focused on repeated multiplication in the exponentiation. Normally, an exponentiation is a shorthand notation for the number of times a number is multiplied by itself. For example, 3^5 means 3.3.3.3.3. When students misapplied this notational form, they reached 5.5.5 for 3^5. In all questions, they used same procedure. One student's written explanations and solutions to Q4 and Q5 are given in Figure 3 revealing that an under-developed conception about repeated multiplication is insufficient to account for the exponentiation process.



Translation of written explanations
I calculated the value for each number and found the answer.

I found the value of the exponents [in the expressions] and found the answer.

Figure 3. Example responses from the students to Q4 and Q5

Obstacles in Understanding of Zero Exponent

Zero is an Identity Element

As shown in Figure 1, 66% of students provided various erroneous responses to Q1 in the written tasks. 14.5% (n = 24) of the students who incorrectly answered Q1 claimed that zero exponent was an identity element although zero is an identity element for addition ($x + 0 = 0 \; or \; 0 + x = 0$). An extract from an interview illustrated how the student reached incorrect result for Q1 in addition to the written explanations in the task.

Sıla: Zero is an identity element and it does not change the result. Thus, I found that $-3^0 = -3$.
Researcher: Can you give me a different example to explain why does not zero have an effect on the number?
Sıla: For example, $100^0 = 100$ and $10000^0 = 10000$.
Researcher: Ok, What is the difference between -3^0 and 3^0?
Sıla: Both have the same result. Since the exponent is zero, it not necessary to make the computation.

Such expressions indicated that the students focused on the identity element under addition and thought that zero exponent equals to the expression itself. When asked about the difference between -3^0 and 3^0, the student replied "both have the same result". Thus, improper use of familiar images which they used for addition in the natural, integer, and real number contexts became an obstacle to the understanding of the zero exponent. As a result, they were unable to recognize that the additive identity (zero) has different meaning in an exponentiation.

Zero is an Absorbing Element

6.6% of the students (n = 11) proposed that if there is zero exponent, the result always becomes zero. One student wrote "the result always is zero if there is zero in exponent. We learned $3 \,.\, 0 = 0$ and $\frac{0}{3} = 0$. Similarly, $-3^0 = 0$." Another student wrote "the result of combining zero with any number is the zero itself." Another two students' written explanations are presented in Figure 4.

Açıklama = 0 her zaman değeri olmadığı için cevabımı bu şekilde yaptım.	*Translation of written explanations* I found this answer because zero never has a value.
Sıfır herşeyi yok eder. O yüzden üstünde sıfır olan sayıların sonucu sıfırdır.	*Translation of written explanations* Zero erases everything. So, the result of the numbers with a zero exponent is zero.

Figure 4. Two students' written explanations for Q1

In Figure 4, the students focused on the absorbing element of multiplication operation. In their explanations, the first student thought that "zero has no value", and the second student stated that "zero absorbs every number. For this reason, if there is zero exponent, the result always becomes zero." Thus, they overgeneralized the properties of multiplication of whole numbers to exponentiation. This shows that if learners are not able to assimilate new knowledge into what they already know then they invent and overgeneralize cognitive structures which are disconnected from the problem without considering the situations in new context.

Expressions including Zero Exponents cannot have a Negative Result

For Q1, 30.9% of students (n = 51) considered that -3^0 equal to 1. In the written explanations, they commonly interpreted -3^0 as "… I remembered that zero exponent is interesting and always results as 1" or "when we find the zero exponent, we always make a division like $2^3 = 8, 2^2 = 4, 2^1 = 2, and\ 2^0 = 1$ For this reason, we always reach 1." In the interviews, to obtain details of their understanding, the students were asked how they found the result of 1 instead of -1as shown in the following dialogue between the researcher and one student:

Researcher: How did you get the value of 1?
Cem: Zero is important. The base -3 raised to the power of zero is equal to 1.
Researcher: Why not -1?

Cem: Negativity in the base is not important if there is zero in the exponent.
Researcher: Could you compare the results of -3^0 as $(-3)^0$?
Cem: We can write $-3^0 = (-3)^0 = 1$.
Researcher: Does it not matter whether the minus sign is inside or outside the parentheses?
Cem: No. The zero exponent transforms each number to 1. You don't need parentheses.

This dialog shows that the student treated -3^0 as $(-3)^0$ and made a notational error. He only concentrated on the zero exponent without considering the exponential expression as a whole. This indicated there is a misunderstanding about the role of parentheses in an exponentiation. Thus, he obtained the value of 1 instead of the value of -1 because he interpreted negative sign as attached to the base. Similarly, during the interviews, another student who made same error in Q1 also considered -3 as an inseparable signed number, instead of recognizing the unary operation. In conclusion, the similarities and differences between the relationship $-n^0 \neq (-n)^0$ were not yet understood by the students. They could not comprehend that if there are no parentheses, exponentiation does not cover the negative sign at the beginning of the number.

Obstacles in Understanding of Negative Exponents

Negatives Become Positive with Even Numbers

An exponential expression having positive integer base and exponent is a prototypical example used at the beginning of the instruction of exponentiation. The written expressions in the task showed that 73.3% of the students correctly found the value of 2^4 in Q3 by making repeated multiplication due to the familiarity of prototypical exponential expressions. However, their prototypical view of exponents did not assist them in handling the negative exponents. Normally, a negative exponent means taking the reciprocal, –which never changes the sign of the result, but in Q2, students (13.9%) claimed that 2^{-4} equals 2.2.2.2 $= 2^4$, instead of $\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{2^4}$. Although these students knew the meaning of x^n as x raised to the n^{th} power, they could not obtain the correct value being affected by the prototype extensions, which is reflected in the following extract from an interview with one of the participants:

Researcher: Why do you think the values of 2^4 and 2^{-4} are equal?
Alya: The minus sign is not important because the power is an even number. I remember that an even exponent always gives a positive result.
Researcher: What do you think about the value of 2^{-4} and -2^4?
Alya: They have the same value if the exponents are even numbers. Both are the same. $2^{-4} = (-2).(-2).(-2).(-2) = 16$.

Alya's comments indicated that she overgeneralized some properties of positive exponents to the properties of negative ones. In other words, instead of finding the equality of a^b and $(-a)^b$ in the case in which b is an even number, she concluded that the values of a^{-b} and a^b are equal in case b which is an even number. On the other hand, based on the written tasks, almost half of the students who incorrectly answered the question of 2^{-4} had inadequate knowledge about the negative exponent. As a result, they neglected the minus sign and obtained 16 as the value of 2^{-4}. One of the students wrote, "I do not know [the meaning of] the minus sign in exponent exactly. In my opinion, the minus sign is not important since we learned $(-)$ becomes $(+)$ if there is an even number.". Similar errors regarding negative exponents were also detected in some questions that included operations with exponents. For example, in Q6, some students obtained the value of $25x625$ as an answer to $5^{-2}.5^4$ (16.5%). Similarly, 30% of the participants found $81x256$ when answering the question of $3^{-4}.4^{-4}$ in Q8 based on misapplication of the relations between odd/even numbers and positivity/negativity of exponents.

A Negative and a Positive make a Negative

In Q2, 39 students (23.6%) found the value of -16 instead of $\frac{1}{16}$. They gave similar reasons why they put a minus sign in front of the base. For example, one student wrote "In Q2, there is a negative and a positive. We multiply numbers in exponents repeatedly. Multiplication of a negative number with positive numbers results negative." Similarly, in an extract from another interview, a participant stated:

Rana: I found -16 because $2^{-4} = -(2.2.2.2)$. In other words, the exponent of a number says how many times to use that number in a multiplication.
Researcher: How did you decide whether your response would be positive or negative?

Rana: There is a negative and a positive. You can get a negative result. $+(-5) = -5$. Similarly, $2^{-4} = -2^4 = -16$

Such students did not know the meaning of negative in power of exponents although they calculated a^b (b = positive integers) correctly. They could not establish a relationship between sign meaning of negatives and reciprocal meaning of negatives in exponents. This situation might be related to their prototypical understanding of integers and natural numbers. Under the influence of prototypical examples, the participants overgeneralized the properties of minus sign in integers to the properties of the minus sign of power in exponent numbers. Furthermore, they commonly committed similar errors when calculating the value of $5^{-2}.5^4$ in Q6 and $5^{-4}.6^4$ in Q7. Of the students, 16% incorrectly calculated $5^{-2}.5^4$ as $-5^2.5^4$ and 20% found the value of $-5^4.6^4$ in Q7 instead of $5^{-4}.6^4$. In conclusion, many students chose to conduct the exponentiation as repeated multiplication with natural numbers even when calculating the value of a negative exponent, rather than finding the reciprocal. Instead, they put a minus sign in front of the base due to inadequate formal knowledge about the meaning of negative exponent. Namely, the idea "a positive and a negative make a negative" can be applied in the multiplication of integers. However, it can become an obstacle to learning when, for example, students were faced with a problem of the form $(+2)^{-4}$.

The Negative Exponent Requires Flipping the Numbers

The negative sign of exponent in an expression is a signal to form the reciprocal. However, a few students misinterpreted this situation. They recognized that a reciprocal was involved but applied the concept inappropriately. For example, they found $2^{\frac{1}{4}}$ for 2^{-4} in Q2. Similarly, they found $3^{-4}.4^{-4}$ as $3^{\frac{1}{4}}.4^{\frac{1}{4}} = 12^{\frac{1}{4}}$. One student said, "If we see a negative in exponent, we must flip this number". Another student wrote, "negative indicates reciprocal; for example, $5^{-1} = \frac{1}{5}$ ". Their written explanations revealed that they focused on the reciprocal of a specific non-zero integer exponent, such as $a^{-1} = \frac{1}{a}$, rather than focusing on a multiplicative inverse like $a^{-b} = (a^b)^{-1}$. As a result, they did not conceptualize $a^{-b} = (a^b)^{-1}$. Instead, their approach to $a^{-1} = \frac{1}{a}$ failed in the calculation of $a^{-b}, (b \neq 0)$. Thus, they concluded incorrectly that $a^{-b} = a^{\frac{1}{b}}$.

Additive and Multiplicative Structures as an Obstacle in the Operations of Exponents

In Figure 1, the percentages of the students' incorrect responses to Q4 (65.5%) and Q5 (76.4%) indicated that they had difficulties with the addition of exponents. The students' common erroneous responses related to operations with exponents are presented in Table 4. In Q4, while some students (18.1%) added exponents in the expressions, some (7.2%) added bases instead of multiplying 3 to 3^5. On the other hand, 10% who incorrectly responded to Q4 added both the exponents and the bases in the expressions. When asked how they obtained 3^{15} or $(9)^5$ when finding the value of $3^5 + 3^5 + 3^5$, they provided similar explanations as follows:

Student 1: When adding exponential expressions, I added exponents because they have the same bases. I remembered that if they have the same bases, we can only add exponents.

Student 2: Since there is addition I added all bases. I did not add the exponents because addition does not influence the exponent parts. So, the result is $(9)^5$.

Students' example comments showed that they used the idea "add exponents if the bases are same" for the addition of exponential expressions. However, this idea refers to the multiplication of the expressions having the same bases, and for this reason, they performed the operation incorrectly. Similarly, 6.6% of the students overgeneralized addition operation to both the bases and powers of exponential expressions. They obtained the value of $(9)^{15}$, instead of 3^6. Furthermore, as seen in Table 4, they exhibited similar errors when calculating the exponential expressions having the same bases with different powers in Q5.

The students made various overgeneralization errors when engaging in the multiplication and division of exponents due to the under-developed conceptions related to additive and multiplicative structures in the algorithms of exponents. Specifically, in Q6 that involved an expression with the same bases and different exponents ($5^{-2}.5^4$ =?), some students (8.4%) tended to multiply exponents instead of adding them (e.g., $5^{-2.4} = 5^{\pm 8}$). Some of the students' written examples are given in Table 3.

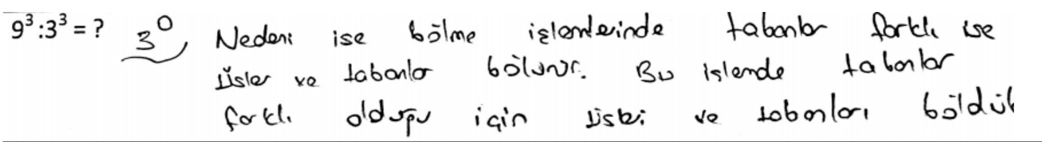
Table 3. Examples of students' written explanations for Q6

Examples responses for Q6	Examples of students' written explanations for Q6
$5^{-2}.5^{4}=?\ 5^{-8}$	In the expressions having the same bases, the exponents are written the same. Necessary operation is done for the exponents.
Sonuç $=5^{-8}$	I wrote the same number at the base because they have the same bases. I conducted multiplication at the exponent part. There is a negative and a positive. So, the result must be negative.

On the other hand, a few students (3%) multiplied both the bases and the exponents in the expressions. Similarly, some students (3%) added the exponents after multiplying the bases. Although these students decided to add the exponents, they incorrectly multiplied the bases (e.g., $5^{-2}.5^{4}=(5.5)^{4\pm2}$). These errors indicated that they overgeneralized some of the algorithmic rules of the multiplication of integers to the algorithmic rules of the multiplication of exponent expressions. In the case of exponent expressions with different bases, students exhibited some additional errors. For example, 16.9% of the students multiplied the bases without considering the difference between the exponents (e.g., finding $30^{\pm4}$ for $5^{-4}.6^{4}=?$). Finally, some students (3%) overgeneralized product rule with the same base in Q7 [e.g., $\mathrm{a}^{\mathrm{m}}.\mathrm{b}^{\mathrm{n}}=(\mathrm{a}+\mathrm{b})^{\mathrm{m}+\mathrm{n}}$]. In this regard, they added both the bases and the exponents when multiplying two exponential expressions although the question involves expressions with different bases and different exponents. In conclusion, their comments on the task indicated that they were confused with basic rules of the multiplication of exponent expressions.

Common errors regarding the division of exponents are also presented in Table 4. When dividing the exponential expressions with the same bases in Q9, approximately 13% of the students tended either to divide the bases or to divide the exponents instead of subtracting the exponents by considering reciprocal form of negative exponents. Their explanations in written documents revealed why they divided the powers. For instance, one of the students commented, "I calculated the result of $5^{6}/\,5^{-3}=?$ as 5^{-2}. It is easy. They have the same bases. For this reason, I kept the bases then I divided exponents due to the division operation." It could be inferred that the student had inadequate knowledge of the division rule in exponentiation and divided the exponents as in the division of integers. In the interview, when asked what he thought about the minus sign of exponent in $5^{6}/\,5^{-3}=?$ in case of division, the student explained his reasoning as: "I know that if we divide a negative and positive number, the sign of the value must be negative." On the other hand, for Q8, seven students multiplied the exponents by keeping the bases. In the written explanations, some students stated, "In division, the sign of the exponent turned from negative to positive. Then, I multiply exponents. From the lessons, I remembered that division means multiplication."

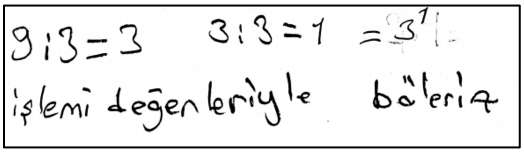
In terms of the division of exponents with the same power in Q10, the students provided various responses, as illustrated in Table 4. For example, some students (4.8%) implied that they just needed to multiply the bases by keeping the exponents. One student wrote, "In the division operation, if the exponents are the same, we must multiply the bases". In this sense, she thought that division required multiplication and obtained the value of 27^{3} instead of 3^{3}. In contrast, seven students (4.2%) subtracted the bases and obtained the value of $9^{3}/3^{3}=(9-3)^{3}=6^{3}$. Furthermore, 8 students (4.8%) subtracted the exponents and divided the bases among themselves. In Figure 5, for Q10, one student wrote, "In the division operation, if the bases are different, we can divide the numbers at bases and exponents among themselves. In the division of exponential expression, we can subtract the numbers. So, the answer is $(9/3)^{3-3}=3^{0}$." In the interviews, another student explained the reason as follows, "In division, I remembered that we should subtract the exponents." When asked what strategies they used in dividing exponents, they stated that it does not matter if the bases are the same or not in the division of exponents. In each case, they stated that it is necessary to subtract the exponents, indicating that they misinterpreted the division rule in exponentiation with the same exponents.



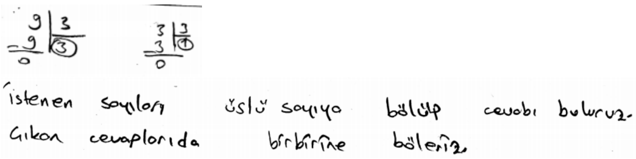
Translation of student's written explanations
In the division operation, if the bases are different, we can divide the numbers at bases and exponents among themselves. In the division of exponential expression, we can subtract the numbers.

Figure 5. One student's written explanation for Q10

For similar reasons, seven students (4.2%) tended to subtract the bases by keeping the exponent. As a result, they found the value of $(9-3)^{3}=6^{3}$ for Q10. One student wrote that "If the exponents are the same, we can subtract the bases. In division, I remember that it is necessary to make subtraction.” Finally, a further seven students (4.2%) divided both the bases and exponents with each other and found $(9/3)^{3/3} = 3^1 = 3$ in Q10. Two students’ responses and written explanations for Q10 are given in Figure 6. Furthermore, in the interviews, when asked how they can find the value of $8^3/3^3$, they proposed that the value becomes $8/3$ and it equals a decimal number. Based on the comments in the interviews and the task, it was concluded that they focused on the division of the bases and the exponents without thinking about the formal meaning of the division operation in exponent expressions.



Translation of student’s written explanations
We divide the expressions by their values.



We can find the result by dividing the bases by the exponents. Then, we divide these values again by themselves.

Figure 6. Two students’ responses and written explanations for Q10

Table 4. Common errors in operations of integer exponent expressions

Operations with exponents		Questions		Students’ common erroneous responses[1]	Representative examples	%	Number of students
Addition operation	Adding exponential expressions with the same exponents and bases	Q4	$3^5+3^5+3^5=?$	Adding exponents	3^{15}	18.1%	30
				Adding bases	9^5	7.2%	12
				Adding exponents & adding bases	9^{15}	6.6%	11
	Adding exponential expressions with different exponents and same bases	Q5	$2^4+2^{-2}+2^3=?$	Adding exponents	2^9 *or* 2^5	24.8%	41
				Adding exponents & adding bases	6^9 *or* 6^5	3.6%	6
Multiplication operation	Multiplying exponential expressions with the same base	Q6	$5^{-2}.5^4=?$	Multiplying exponents	5^{-8} *or* 5^8	8.4%	14
				Multiplying bases & adding/subtracting exponents	$25^{\pm2}$ *or* $25^{\pm6}$	3%	5
				Multiplying exponents &multiplying bases	25^{-8} *or* 25^8	3%	5
	Multiplying expressions with different bases and exponents	Q7	$5^{-4}.6^4=?$	Multiplying bases	30^{-4} *or* 30^{+4}	16.9%	28
				Multiplying bases & adding exponents	30^0 *or* $30^{\pm8}$	7.8%	13
				Adding bases & adding/subtracting exponents	11^0 *or* $11^{\pm8}$	3%	5
	Multiplying exponential expressions with the same exponents	Q8	$3^{-4}.4^{-4}=?$	Multiplying powers	12^{-16} *or* 12^{16}	3.6%	6
				Adding/subtracting exponents	12^0 *or* $12^{\pm8}$	4.8%	8
				Adding bases & adding/subtracting exponents	7^0 *or* $7^{\pm8}$	4.8%	8
Division operation	Dividing exponential expressions with the same bases	Q9	$5^6/5^{-3}=?$	Division of exponents	5^{-2} *or* 5^2	13%	11
				Multiplication of exponents	5^{18} *or* 5^{-18}	4.2%	7
				Division of bases	1^6 *or* 1^{-3}	3%	5
	Dividing exponential expressions with the same exponents	Q10	$9^3/3^3=?$	Multiplication of bases	27^3	4.8%	8
				Subtraction of exponents	$(9/3)^{3-3}=3^0$	4.8%	8
				Subtraction of bases	$(9-3)^3=6^3$	4.2%	7
				Division of exponents	$(9/3)^{3:3}=3^1$	4.2%	7

[1]Note: Students’ errors originating from the negativity of numbers are not given in this table as they were mentioned in reference to obstacles related to the minus sign of exponents. The entries in the table are confined to their incorrect responses and reasons originating from operations with exponential expressions.

Conclusion and Discussion

This study examined eighth-grade students' achievement in dealing with integer exponents, how they responded to the questions and the reasons for their failure to correctly respond to some questions. Specifically, the obstacles related to the questions concerning integer exponents were identified. The findings of the study not only confirmed the results of previous studies, but also moved the discussions one step ahead by providing a deeper examination of student understanding of exponents. The results indicated that many students failed to answer most of the questions in the task, which is consistent with the previous research studies on learners' understanding of exponents (Cangelosi et al., 2013; Pitta-Pantazi et al., 2007; Rabin et al., 2013).

In line with the second research question of the study, the findings revealed that *multiplication in the definition of exponentiation* was an obstacle to the understanding of the meaning of exponentiation. Normally, the exponent "b" in the expression a^b represents a^b repeatedly multiplying a by b times. However, some students participating in the current study multiplied the base and the exponent in all questions, but other students computed a^b repeatedly multiplying b by a times. These results revealed that some students misinterpreted the meaning of (repeated) multiplication in the definition of exponentiation, such as $a^b = a.b$ or $a^b = \underbrace{b.\dots.b}_{a\ times}$. These findings have some similarities with the studies conducted in similar topics. For example, MacGregor and Stacey (1997) investigated students' understanding of algebraic notations concluding that students' persistent misuse of exponential notation suggests an insecure foundation of the concepts of multiplication, repeated addition, and repeated multiplication, such as treating $x + x + x + x$ as 4^x or x^4, instead of $4x$. From this point of view, students' inadequate knowledge about exponentiation causes problems in other mathematical contexts. In another example, Banerjee (2011) proposed that students failed to understand repeated multiplication in algebraic expressions due to confusion with notational confusion between repeated addition and repeated multiplication [e.g., $(2 + 3)^2 = 2.(2 + 3)$]. On the other hand, Zazkis (1998) found that students produce incorrect responses about odd and even numbers due to the implicit identification of 3^{99} with $3x99$. In her study, students interpreted 3^{100} as an even number. Similarly, in Weber's (2002) study about college students' understanding exponents and logarithms, some students thought that 5^{14} would be an even number because they believed an odd number raised to an even power would be even.

Students' difficulty with zero exponents is one of the most frequently reported incorrect reasoning for questions concerning exponent expressions (Cangelosi et al., 2013; Kontorovich, 2016; Levenson, 2012). The current study enriches the existing literature about zero exponents since it found that *absorbing element of multiplication operation* and *identity element of addition operation* are the obstacles to students' understanding about zero exponents. In other words, some students tended to use the identity element of addition or the absorbing element of multiplication over exponent expressions, and made overgeneralization errors. In such a situation, while some students thought $-3^0 = -3$ by proposing zero does not influence the value of exponential expressions as $-3 + 0 = -3$, others reached $-3^0 = 0$ as in $-3.0 = -3$. This result has similarities to the results of studies in which researchers concluded that students misinterpret the identity element of addition, zero, and think that zero exponents equals to the base itself (Cengiz, 2006; Crider, 1998). The reason for such thinking was related to their inability to connect new mathematical ideas to their existing knowledge by making the correct modifications (Brousseau, 2002; Levenson et al., 2007; Rabin et al., 2013). Levenson (2012) observed that mathematics teachers have insufficient knowledge about the formal definition of $a^0 = 1$, which indicates that what the teacher knows about the concept and how s/he teaches it in the classroom affect students' understanding of exponents. Another obstacle the students in the current study faced concerning the zero exponent was *expressions including zero exponents cannot have negative result.* Some students interpreted this as expressions including zero exponents cannot have negative result. They did not give a meaning to the minus sign in the expression due to the presence of the zero exponent. As a result, the students thought that $(-3)^0$ and -3^0 were the same. Based on Sfard's (1991) theoretical model about the dual nature of mathematical conceptions, such as structural and operational, Cangelosi et al. (2013) concluded that students did not distinguish between $(-n)^2$ and $-n^2$ because they did not understand difference and similarities between them.

The higher percentage of correct answers in the third question was related to the nature of the exponent number since it is a prototypical example having positive base and exponent. In the literature, prototypical examples are accepted to be easier to understand than the other examples (Hershkowitz, 1990; Vinner & Hershkowitz, 1983). For this reason, they served as the "reference cognitive point" (Pitta-Pantazi et al., 2007; Schwarz & Hershkowitz, 1999) for the computation of exponential expressions. Thus, the relatively higher percentage of correct responses to Q3 can be attributed to the overutilization of prototypical examples at the beginning of the teaching process of exponentiation. In this way, students understand exponents in x^n where *n* are a positive integer. However, the prototype extensions to exponent expressions with negative exponents produced an obstacle in the students' understanding of exponents, when they do not have a meaningful understanding of negative numbers. In this study, obstacles originated from negativity were grouped into three categories. First, *negatives becoming positive with even numbers* was an obstacle in the study. In this sense, some student overgeneralized the rule as being valid in positive even integer exponents. In other words, instead of $a^b = (-a)^b$ in case $b\ (\neq 0)$ is an even number, they concluded that the values of a^{-b} and a^b are equal in the case where $b\ (\neq 0)$ is an even number. Second, *a negative and a positive make a negative* was found to be another obstacle in the understanding of negative exponents. Normally, *a negative and a positive make a negative* can be applied in the multiplication of integers. However, students could not distinguish the unary/binary meaning and reciprocal meaning of negative sign. As a result, they obtained $a^{-b} = -a^b, (b \neq 0)$ incorrectly. Thus, the students could not understand that the additive inverse (negative) becomes the multiplicative inverse (reciprocal) in exponents. In conclusion, the first and second obstacles are related to the students' conceptions based on the definition of exponentiation as repeated multiplication with natural numbers. Some researchers (Confrey & Smith, 1995, Davis, 2009; Ellis et al., 2016; Lakoff & Nunez, 2000; Presmeg & Nenduardu, 2005; Weber, 2002) have pointed out that the conception of exponential operation based on repeated multiplication with natural numbers is inadequate to perform operations with non-natural exponents and logarithms and to appropriately generalize rules, such as the multiplication and power properties of exponents. For instance, students who only interpret exponents as repeated multiplication, expressions, such as 2^{-4} or $2^{\frac{2}{3}}$ will be difficult to understand from a repeated multiplication perspective (Weber, 2002). In another study, Rabin et al. (2013) conclude that students' understanding of exponents involves multiplication rather than division. Thus, students consider that any computational formula for exponents should involve multiplication rather than division due to the repeated multiplication definition.

The reciprocal of a number denotes the inverse of that number with respect to multiplication [e.g., $(2^4)^{-1} = 2^{-4} = \frac{1}{2^4}$]. In this study, a few students who were aware of the difference between negative and positive exponents did misinterpret the negative sign in exponents. At this point, *the negative exponent requires flipping the numbers* became an obstacle in students' understanding of negative exponents. The students concentrated on the reciprocal of a specific number such as $a^{-1} = \frac{1}{a}$ rather than focusing on the multiplicative inverse like $a^{-b} = (a^b)^{-1}$ and they found the incorrect result of $a^{-b} = a^{\frac{1}{b}}$. Cangelosi et al. (2013) called such errors as *the roaming reciprocal*. According to the researchers, *language* and *notation* were the main reasons in students' errors. They argued that "students who incorrectly simplified the expression 2^{-3} appeared to have a rudimentary operational understanding of multiplicative inverse linked to the term flipping. The appearance of the negative sign was a signal for them to form a reciprocal, but it was unclear to them what to flip (p. 78)." Furthermore, students' conceptions of the definition, as $a^{-1} = \frac{1}{a}$, failed to convey the notion of the multiplicative inverse ($(2^4)^{-1} = 2^{-4}$). On the other hand, some researchers also interpreted students' errors concerning the reciprocal linked with the APOS theory of Dubinsky (1991). For example, based on the theory, Weber (2002) explained that students need to understand a^b as an object in order to find a^{-b} correctly.

There are a limited number of studies on how students performed the algorithms of exponents (Cengiz, 2006; Crider, 1998). For example, Cengiz (2006) used complicated procedural questions in which it is difficult to determine why learners incorrectly performed the operational algorithms for exponents. So, the current study tries to clarify the obstacles hindering students' performance on the operations with exponents. In this study, under-developed conceptions about additive and multiplicative structures in the algorithms of exponents were found to be an obstacle especially in the questions related to operations of exponents (Cangelosi et al., 2013). Students used multiplication and division in the operation of exponential expressions instead of addition and subtraction or vice versa. As a final point, although the scope of this study includes only determining the obstacles and not how to overcome those obstacles, some implications and suggestions for future studies can be presented. For example, in order to prevent the creation of obstacles and remedy the existing obstacles in the misuse of multiplication in the definition of a^b, teachers can promote a discussion environment in which the students can explore the meaning of a^b by using alternative approaches to exponential growth, such as quantitative reasoning and covariation (Ellis et al., 2015). To avoid obstacles regarding zero exponents, students might be recommended to use either a decreasing geometric sequence or to preserve the division rule as two instructional strategies for presenting zero exponents (Levenson, 2012). However, Rabin et al. (2013) provided evidence that decreasing geometric sequence does not convince all students although they can see the pattern between numbers. The ability of teachers to make the stated recommendations requires well-structured knowledge of the subject matter and the pedagogical content. Namely, the teacher's knowledge of content becomes crucial since this affects the way in which they represent the nature of knowledge within the area of content to their students (Ball, 1990; Even, 1993). From this perspective, teachers are responsible for presenting the students with correct rules and mathematical explanations to memorize such as such as "An exponentiation is multiplication of numbers" and "A negative and a positive make a negative". If student do not conceptualize the meaning of such statements, they misapply them due to the vague and lacking nature of these statements. Furthermore, primarily prospective teachers' knowledge of exponents should be explored. In this regard, the discussion of obstacles presented in the current study is useful in the preparation of prospective teachers, especially in the teaching method courses. From another perspective, it might be recommended that researchers investigate and develop teachers' definitional knowledge about the meaning of zero exponent and superscript of "(-)" using detailed methodological approaches, such as teaching experiment methodologies and design-based research. Finally, the observations of teachers during the teaching of exponents would provide opportunities to determine the origins of each obstacle and to develop efficient ways for solving problematic situations.

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