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Can Classification Criteria Constitute a Correct Mathematical Definition? Pre-service and In-service Teachers' Perspectives

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Abstract

The study reported here addresses pre-service and in-service teachers' attitudes toward mathematical-geometrical definitions. The goal of the study is to investigate whether understanding the role of definitions as classification and identification criteria will guarantee that participants: (1) accept that there may be more than one equivalent definition for particular concept; and (2) accept the minimal definitions which include necessary and sufficient attributes to be legal definitions. Fifty-three math educators participated the study, including 22 pre-service junior teachers, 19 pre-service senior teachers and 12 in-service senior teachers. The findings indicate that considering (an) attribute/s as sufficient in order to classify examples and non-examples of the concept did not guarantee considering this/ese attributes as concept definition. 56% of the participants did not accept equivalent definitions as legal definitions, 36% of them (the participants) did not accept the minimal definitions, which include necessary, and sufficient attributes to be a legal definition. For many participants, the essence of the mathematical concept is more important than the essence of the mathematical definition.

Introduction

Vinner & Hershkowitz (1980) and Tall & Vinner (1981), focused on the cognitive construction of mathematical concepts, and proposed a model of two components: the concept definition - the verbal description of the mathematical concept, which characterizes the concept mathematically, and the concept image - the cognitive structure that includes all the examples and the processes related to the concept in the learner's mind. Many studies investigated the role of defining in mathematics education (e.g. Borasi, 1992; Leikin & Winicky-Landman, 2001; Mariotti & Fischbein, 1997; Moore, 1990; Weber, 2002). The research literature includes many studies on the ability to define in mathematics, especially about geometrical concepts. These studies found that students, pre-service and in-service teachers all of them have difficulty to define and to understand mathematical-geometrical definitions (e.g. Fujita & Jones, 2007; Hershkowitz, 1987; Marchis, 2012; Pickering, 2007). Few studies, however, dealt with the linkage between understanding the role of mathematical defining and the perception of mathematical-geometrical definitions (Van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). The current study attempt to that, more explicitly the aim of this study is to examine whether the attitudes of students and teachers toward the definitions' role affect their understanding and perception about mathematical-geometrical definitions.

Background

One of the main roles of mathematical defining is creating uniformity about the meaning and the essence of the concepts, which allows the mathematician community to engage and communicate in a simple and easy way (Borasi, 1992; Leikin & Winicky-Landman, 2001). Van Dormolen and Zaslavsky (2003) emphasized the role of the mathematical definition within mathematics as a formal science. They defined the features required for a mathematical definition to be a good definition, one of these futures is that the definition corresponds to the deductive system to which it belongs and is a basic part of it. Vinner (1991) makes five assumptions about mathematical definitions: concepts acquired by their definitions; definitions must be minimal; definitions must be elegant; definitions are arbitrary; and students use definitions to solve problems. Critical condition for performing mathematical tasks is relying on formal definitions. These mathematical tasks might be identification and classification of examples and non-examples of a particular concept, the tasks of devising a problem-solving strategy and the tasks of constructing a mathematical proof (Moore, 1994; Vinner, 1991; Weber, 2002). Ouvrier-Buffet (2006) referred to what she calls the situation of definition construction (SDC), and argued that definitions allow us to work in scientific processes, and she claims that SDC(s) give(s) us an opportunity to work on scientific processes (construction of definitions and proofs in particular). She describe the components of these scientific processes and in her words:

> *Scientific processes are constituted by students' experiments with different cognitive attitudes: doubting, conjecturing, refuting, generating new counter-examples, testing etc..."* (Ouvrier-Buffet, 2006, p. 279).

Mariotti and Fischbein (1997) discussed another role of mathematical definition, that it helps us to understand the essence and the meaning of the mathematical concepts. They argue that defining was the basic problem of mathematics education, they claim*: "First of all, objects which we are dealing with must be stated and clearly defined… defining is a basic component of geometrical knowledge and learning to define is a basic problem of mathematics education"* (Mariotti & Fischbein, 1997, p. 219).

The van-Hiele theory (1958), is the most comprehensive theory concerning teaching and learning geometry. According to van Hiele & van Hiele's theory, the third level is the non-formal deduction (Ordering): at this level, the learner can construct internal connections between the different properties of the same shape and connections between the properties of different shapes. The learner understands the importance of correct definitions, understand how a particular attribute derives from another attributes, and the inclusion relationship between groups of shapes. At the fourth level, the learner understands the role of definitions and recognizes the characteristics of a formal definition, such as necessary and sufficient attributes in the definitions and equivalence of definitions.

In the study of geometry, it is acceptable that the definitions include the minimum subset of necessary and sufficient attributes. The necessary attributes are those which exist in all the examples of the concept, whereas the sufficient attributes are a subset of necessary attributes, which allow us to deduce the remaining necessary attributes. Vinner (1991) and Van Dormolen & Zaslavsky (2003) noted that it is preferable that mathematical definitions include a minimum number of necessary attributes sufficient to define the concept. They added that the definitions must be elegant, that is, when selecting a definition out of two equivalent definitions, we should chose the one that uses fewer words and symbols or that "looks" better. Leikin and Winicky-Landman also emphasized the need for minimal definitions, in their words *"The essence of a mathematical definition of any form is the constitution of necessary and sufficient conditions of the defined concept and the "minimum" of the set of these conditions"* (Leikin & Winicky-Landman, 2000, p. 64).

Freudenthal (1973) distinguishes between two types of mathematical definitions. The first is creative definition; this "builds" new objects out of familiar objects. Creative or constructive definitions are done by adding, modifying, distinguishing or including attributes to previous concepts on which they are based. The second is descriptive definition that outlines a known object by singling out a few characteristic properties. A descriptive definition will be perfect when selecting an appropriate subset of attributes out of all the attributes of the concept that is sufficient to capture the other attributes. When singling out a few necessary and sufficient attributes, the descriptive definition can be combined with a creative definition. For example when we define the rectangle as "quadrilateral which its diagonals crossing each other and equal each other", this definition is combined with the creative definition of the rectangle as "parallelogram which it's diagonal equal each other" because quadrilateral which its diagonals crossing each other is parallelogram.

Difficulties with Definitions

Linchevsky, Vinner & Karsenty (1992) and de Villiers (1998) reported about the tendency of students and preservice teachers to mention a long list of all the attributes of the concept. In fact, these long descriptive definitions are correct, but many mathematic educators claim that it is preferable that mathematical definition to contain minimal attributes and to be elegant or parsimonious as indicated above (e.g., Vinner, 1991; Van Dormolen & Zaslavsky, 2003; Leikin & Winicky-Landman, 2001). On the other hand, there are those who in certain cases also prefer a non-minimal definition (de Villiers, 1998; Pimm, 1993; van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). For example, it is possible to define two similar triangles as "two triangles are similar if two angles of one triangle are congruent to the corresponding two angles of another triangle"; from these attributes, we can deduce the four remaining attributes about the lengths of corresponding sides which are proportional. From the pedagogically perspective adherence only to this minimal definition, may impair the perception and the essence of the similar triangles concept (Zaslavsky & Shir, 2005).

One characteristic of mathematical definitions appreciated by mathematicians and mathematical educators is that for a given mathematical definition might be other equivalent definitions of the same concept (Harel, Selden & Selden, 2006). Leikin and Winicky-Landman (2001) researched the understanding of mathematical definitions among high school teachers in a non-geometric context; they found that many participants did not notice that a particular concept could be defined in a number of equivalent ways. Haj Yahia, Hershkowitz & Dreyfus (2014) found that many senior school students rejected correct geometric proofs because they failed to notice that there might be more than one definition for a particular concept. When we attend to the equivalent geometrical definitions, however, we perform as expected in van Hiele & van Hiele (1958)'s third level: we accept the notion that we can derive one attribute from the other/s. For example, if in a quadrilateral every two opposite sides are parallel, then we can deduce that every two opposite sides are equal and vice versa – and these two statements are equivalent.

This brief review has indicated some difficulties in two abilities: understanding the logical structure of mathematical definitions and recognizing the features of the mathematical definitions. These abilities are some of the abilities expected in the third and the forth levels of van Hiele's whereas additionally the learners expected to understand the essence and the roles of the formal definitions. Johnson et al., (2014) claim that teachers' conceptions of definitions are the basis for their mathematics teaching, especially in geometry teaching. I could not find enough studies that have examined the impact of understanding the essences and the roles of formal mathematical definitions on understanding the features and the logical structure of them (mathematical definition) among all the populations and especially among teachers and pre-service teacher's populations. Therefore, there is a need to do that. To say more explicitly this study attempt to examine whether pre-service and in-service teachers understanding of the roles of mathematical definition, classification, and problem solving will affects their understanding about the features and the logical structure of mathematical definitions. The importance of this research is that if we put the result of it up to the teachers' awareness, it can affect their teaching and the aspects they will emphasize, and as a result lead to deeper conceptualizing of mathematical definitions.

Research Rationale and Goals

In the previous sections I presented studies, which emphasize on the roles of the definition within the formal mathematical system; other researches deal with difficulties concerning mathematical defining. However, I could not find in the literature a clear focusing on the relationships between these two research domains - understanding the roles of geometric concepts definitions and the defining processes related to these concepts. The present research attempts to fill the gap. I take one aspect of the definitions roles- classification and identification criteria, and I investigated whither understanding this aspect will affect the defining process. To say more explicitly, the goal of the study is to investigate whether understanding the role of definitions as classification and identification criteria will guarantee that participants: (1) accept that there may be more than one equivalent definition for particular concept; and (2) accept all minimal definitions which include necessary and sufficient attributes to be legal definitions.

Method

The study reported here addresses pre-service and in-service teachers' conceptions about mathematical-geometrical definitions. It focuses on one of the basic roles of mathematical definition – the identification and classification of examples and non-examples of the concept – and examines whether understanding this basic role will affects and helps the participants to understand the mathematical definition and its logical structure. If learners understand that a particular attribute or set of attributes may be used as a criterion for classifying and identifying examples and non-examples of a particular concept, then they might consider whether this attribute or set of attributes are necessary and sufficient for identification and classification – the main prerequisites of a mathematical definition. Thus, the aim of this study is to examine whether understanding this role of definitions guarantees that participants:

- Accept that for a particular geometric concept might be more than one definition and all of them are equivalent.
- Have a sense about minimal definitions.

I will investigate whether there is differences between attitudes of preservice senior math teachers, pre-service junior math teachers and in-service senior math teachers toward definitions.

Participants

The study included 53 participants, including 22 pre-eservice junior, 19 pre-service senior, and 12 in-service senior math teachers. The pre-service teachers studied in two education colleges. The in-service teacher population available to me included teachers from six-year high schools (seventh through twelfth grade) in the centre of the country. All are consider as good schools in terms of the level of success in matriculation examinations. All teachers had more than seven years' experience in math education and all of them were university or college graduates who trained mathematics teachers. The pre-service teachers were selected out of four groups during one lecture. It was explained to the pre-service teachers that answering the questionnaire did not constitute a factor which affecting their scores. The study goals were explained to all the participants before they started answering the questionnaire.

Instruments and Procedure

The research instruments include a questionnaire (see Appendix 1) which developed especially for this study and semi-structured interviews. The questionnaire examined the participants' conception of the mathematical definition of the parallelogram and similar triangle concepts. It included six tasks. In Tasks 1, 3 & 5, the participants were asked to answer whether a given attribute or set of attributes could be used to classify examples and non-examples of concepts (see Appendix 1). The rest three tasks the participants were asked to answer if the same attribute or set of attributes mentioned in tasks 1, 3 &5 could be definitions for the same concepts: the parallelogram and similar triangle concepts (see tasks 2, 4 & 6).

In Task 1, the answer is that we can use the criterion that two angles in one triangle are identical to two angles of another triangle in order to sort examples and non-examples of similar triangles concept, because we use necessary and sufficient attributes for similar triangles. In Task 2, both definitions are correct, but Sami's is non-minimal. Rami's definition uses necessary and sufficient attributes that lead to all the critical attributes of the concept, to which Sami refers. For Task 3, the answer is yes. In Task 4, Sami's claim about Rami's definition is wrong: his definition is equivalent to Sami's and both could be correct definition for the parallelogram concept. In Task 5, the answer is yes. Finally, the answer in Task 6 is that the both equivalent definitions are correct.

In some of the tasks, the participants were asked to reflect on proposed answers. This gave them the opportunity to use critical thinking. In addition, while students were required to explain their responses, they uncovered some of their views and knowledge regarding definitions. After administering the questionnaire and analysing its results, the author interviewed five participants who provided answers and explanations, which represented the difficulties of the majority of the population. The interview took about seven minutes; the structured part included the same questions that had been asked in the questionnaire, while the unstructured part included questions formulated according to the interviewees' responses. The goal of the interview was to examine whether the participants were indeed certain of their answers and to clarify points that were not addressed by the questionnaire, which required deeper examination. For example, in the questionnaire, I wanted to examine whether the participants accepted a minimal definition of similar triangles that included only angles as a formal definition, and I did not check whether the participants accepted other minimal definitions of the same concept. In the interview, I had the opportunity to do so, thus the interviews adding an important nuance to the questionnaire findings.

I chose to deal with tasks related to the parallelogram and similar triangles concepts because: (1) these concepts are very familiar to the participants, they teach or will teach them in very early ages, almost at the beginning of teaching proofs and deduction. (2) The logical structure that exist between the attributes of the same concepts and this makes the parallelogram and similar triangles easy subjects to illustrate the study. These tasks are only representative tasks; I can design other tasks related to the same concepts or to other concepts. If I talk about tasks related to the same concept, I can choose other couple of tasks for similar triangles concept; in one task, I ask whether three proportional sides could be classification criteria for similar triangles and in the other task, I ask if the same attribute could be a definition for similar triangles concept. An example for tasks related to other concepts; I can choose the rectangle concept and design the couple tasks; in one, I ask whether a quadrilateral with three right angles is a criterion for classification of rectangles rather than non-rectangles and in the other task, I ask whether the same attribute could be a definition of the rectangle concept. I am not aware about previous studies that have examine the same issue.

Findings

In this section, I describe participants' answers in detail, based on an analysis of the three pair's tasks (1&2, 3&4 and 5&6, respectively). This is because in these parallel tasks, the same attribute/s have been used once to examine whether the participants thought that we could use these attributes to classify examples and non-examples of the concept, and another to examine whether the participants thought that we could use the same attributes to define the same concept. I will also examine the differences between the three teacher groups.

Similar Triangles Tasks (1&2)

Table 1 refers to Task 2, where participants were asked to choose between Sami's definition that two triangles are similar if and only if their corresponding angles have the same measure and the lengths of corresponding sides are proportional, and Rami's definition that two triangles are similar if they have two congruent angles. The table shows that about 23% of participants claimed incorrectly that only Sami's non-minimal definition was correct (25% of senior in-service teachers, 16% of senior pre-service teachers and 27% of junior pre-service teachers). Selvi, Yossi and Udi accepted that two triangles, $\triangle ABC$ and $\triangle A'B'C'$, which have two congruent angles were similar. Despite their agreement that two equal angles were sufficient to identify similar triangles they claimed that Rami's definition, based on the same sufficient attribute, was incorrect. Selvi explained, "Rami gave the theorem by which we can show that the triangles are similar triangles". Udi wrote, "Rami use a theorem and not a definition". Finally, Yossi argued that "Rami's words are sufficient conditions for identifying similar triangles, and do not reveal the mathematical essence underlying similar triangles". Selvi and Udi's explanations indicate that they do not understand the essence and the meaning of the theorem; they do not understand that it proves that we do not need all the attributes in Sami's definition, but we can make it differently by mentioning fewer attributes. Yossi demands that the definition have to reveal the mathematical essence of the concept. For Yossi, the equality of angles does not seem to fully reflect the meaning of similar triangles.

About 64% of participants (63% of pre-service junior teachers, 68% of pre-service senior teachers and 58% of senior teachers) answered correctly that both definitions were correct (half of all the participants preferred Rami's minimal definition the rest preferred Sami's non-minimal definition). Some of the participants who preferred Rami's minimal definition explained their answers. Tamir, for example, explained, "Sami's definition derives from Rami's definition"; and Ayal explained, "Rami uses a similarity theorem and this is an accurate definition. Sami describes the meaning of similarity and this is a good but very long definition". These participants behaved as expected in van Hiele and van Hiele's (1958) third level. If we take into account only the participants who claim correctly that both definition are right; from pre-service junior math teachers more participants prefer Sami's non-minimal definition, from pre-service senior math teachers more participants prefer Rami's minimal definition and from in-service senior math teachers more participants prefer Rami's minimal definition.

Tamir and Ayal were certain about their choice. Tamir understood the equivalence of the definitions and Ayal understood that the theorem of similar triangles (angle, angle theorem) gave us a minimal definition for similar triangles. Shahaf, on the other hand, answered that she preferred Rami's minimal definition but was not certain about her answer: "I am hesitant about my answer. Without a doubt Rami is right, it's sufficient that if two angles in one triangle equal to two angles in other triangle in order to say they are similar triangles, but Sami's definition is clearer and yet on the other hand it is also longer and we may not really require all the text. So I'm not sure which is better".

Table 1. Which definition/s of similar triangles is/are correct?

	Only Sami's	Only Rami's	Both. Rami's is preferable	Both. Sami's is preferable	Both are incorrect	Total
Preservice junior math teachers	6 11.32%	2 3.77%	3 5.66%	11 20.75%	0	22 41.51%
Preservice senior math teachers	3 5.66%	3 5.66%	9 16.98%	4 7.55%	0	19 35.85%
Inservice senior math teachers	3 5.66%	1 1.89%	5 9.43%	2 3.77%	1	12 22.64
Total	12 22.64%	6 11.32%	17 32.08%	17 32.08%	1 1.89%	35 100%

Hile, Nofar and Samer accepted that two triangles, △ABC and △A′B′C′, which have two congruent angles were similar. Although, they preferred Sami's non-minimal definition. Samer explained: "Sami's definition is the accepted for the concept of similarity of triangles, with the necessary attributes mentioned in detail, whereas Rami's definition includes the sufficient attributes from which all the rest critical attributes in the accepted definition can be deduced". Samer understood that one definition can be derived from the other, and in that meets the fourth level of van Hiele & van Hiele’s (1958) conceptualization. Samer preferred the definition accepted in the mathematical education community, and the uniformity of the definitions is critical for him. Hile and Nofar preferred the long definition because it contained all the critical attributes. Hile explained: "Sami's definition is a complete definition because it contains both criteria (1. Angles 2. Ratio of the sides). Rami's definition contains the criterion of angles: the three angles of the triangles are equal respectively – it's enough to have two equal angles because we will complete 180 degrees (sum of the angles in a triangle)". Nofar explained, "For similar triangles it's also worth noting the ratio between the sides".

Table 2 shows the percentage, out of all the participants without consideration about the three sup-groups. The table shows that 83% of participants claim correctly that two triangles, which have two congruent angles, can be a criterion of classification of similar triangles. Out of these participants, 25% (20% of the total sample) claim incorrectly that Rami's minimal definition is inaccurate, and thus fail to meet van Hiele & van Hiele’s (1958) fourth level. Another 34% (28% of the total sample) of the claim that the both definition are right and prefer Rami's minimal definition. Finally, 27% among the participants who claim that these attribute of two triangles, which have two congruent angles is a classification criterion for similar triangles (22% of the total sample) still prefer Sami’s non-minimal definition that includes superfluous attributes.

Table 2. Can the classification criterion for similar triangles be a definition?

	Only Sami’s definition is correct	Only Rami's definition is correct	Both are correct. Rami’s is preferable	Both are correct. Sami’s is preferable	Both are incorrect	Total
The criterion classifies similar triangles	11 20.75%	5 9.43%	15 28.3%	12 22.64%	1 1.89%	44 83.02%
The criterion does not classify similar triangles	1 1.89%	1 1.89%	2 3.77%	5 9.43%	0	9 16.96%
Total	12 22.64%	6 11.32%	17 32.08%	17 32.08%	1 1.89%	53 100%

I selected Yossi for interview because his answers and explanations represented the difficulties of the majority of the population concerning the similar triangles tasks. In the interview, he accepted that that the attribute of two equal angles was a classification criterion for similar triangles but claimed incorrectly that Rami's minimal definition was incorrect.

Interview 1: Yossi

Interviewer: Can we use the criterion "two angles of one triangle have the same measure as two angles of another triangle" to sort examples and non-examples of two similar triangle?
Yossi: Yes, we can.
I: In the questionnaire, you claim that Rami’s definition […] is wrong.
Y: Yes.
I: Although it classifies similar triangles?
Y: Yes, because it does not give us the essence of similar triangles.
I: Could the attribute "three sides are proportional in two triangles" be a classification criterion for similar triangles?
Y: Yes, this is the theorem.
I: One student defined similar triangles as follows: "two triangles are similar when all their corresponding sides have lengths in the same ratio". Is that a correct definition?
Y: I can accept it as a correct definition.
I: Why?
Y: Because in this definition the essence of the concept is very clear.
I: Does the [aforementioned] attribute lead to the attribute "two angles of one triangle are equal to two angles of the other triangle"?
Y: Yes.

I: And the converse is also correct?[…]
Y: Yes.
I: Do you want to change your answer above?
Y: No, because of the essence of the concept. In addition, we cannot find the other definitions that you mentioned in the textbooks.

For Yossi, the attribute that could be a criterion for classification is not sufficient in order for it to constitute an accurate, formal mathematical definition. Yossi does not understand that all theorems of similar triangles provide us with economical definitions for similar triangles concept. It is important for him that the definition include the attributes that embody the essence of the concept (the sides are proportional). Yossi accepts that "three sides are proportional in two triangles" and "two angles of one triangle have the same measure as two angles of the other triangle" as criteria for classifying similar triangles. He also accepts that that these criteria are equivalent. However, he claims that only criteria that highlight the essence of the concept can constitute a formal definition. Thus, Yossi fails to meet van Hiele & van Hiele’s (1958) fourth level where the learner understands the function and the features of mathematical definitions. That it in terms of identifying and classifying examples and non-examples of the concept and that it is includes necessary and sufficient attributes, and in terms that for particular mathematical concept might be more than one definition and all of them are equivalent.

Parallelogram Definition (Tasks 3&4)

Recall that Rami defined the parallelogram as a quadrilateral whose diagonals crossed each other. Sami claimed that Rami's definition was incorrect and base on an attribute derived from the definition. Table 3 shows that the vast majority (57%) of the participants answered incorrectly that Sami's claim was correct (66% among senior in-service teachers, 47% among senior pre-service teachers and 60% among junior pre-service teachers).

Nofar and Mile did not accept that "quadrilateral in which the diagonals cross each other" was sufficient for a quadrilateral to be a parallelogram. Mile explained, "The fact that the diagonals cross does not necessarily mean that it is parallelogram". Nofar's explanation was, "One of parallelogram’s attributes is that the diagonals cross each other". Probably Nofar require another attribute for the parallelogram definition, because the crossing attribute was insufficient.

Table 3. Is Sami's definition of a parallelogram correct?

	Yes	No	Other	Total
Preservice junior math teachers	13 24.53%	7 13.2%	2 3.77%	22 41.51%
Preservice senior math teachers	9 16.98%	9 16.98%	1 1.89%	19 35.85%
Inservice senior math teachers	8 15.09%	4 7.55%	0	12 22.64%
Total	30 56.6%	20 37.74%	3 5.66%	53 100%

Ayal, Reem, Silve, Yossi and Udi accepted that "quadrilateral in which the diagonals cross each other" was sufficient for a quadrilateral to be a parallelogram, but claimed that it could not constitute a formal definition. Yossi explained: "Sami is right, Rami's definition is based on an intrinsic attribute of the parallelogram, that is, if this property were to determine the parallelogram’s definition, we would not see in their mind the central aspect of the essence of the parallelogram, since Rami confused between the attribute and the definition". Yossi was consistent in his answers, and once more demanded that the definition include the attribute that highlighted the essence of the parallelogram concept. Again, he failed to meet van Hiele & van Hiele’s (1958) fourth level.

Udi explained his answer as follows: "There is nothing more to explain than what Sami said. What Rami said is a trait that results from the definition". Silve's explanation was, "it's deduced from the definition". Reem's explanation was, "It is impossible to define a concept by its characteristics". Ayal's explanation was, "The attribute of crossing diagonals is common to several shapes and it’s derived from the definition". Reem, Udi, Ayal and Silve distinguished between definitions and attributes, and that the attributes of crossing diagonal were derived from the exclusive parallelogram definition.

Only about 38% of the participants (33% of in-service senior teachers, 47% of pre-service senior teachers and 32% of pre-service junior teachers) answered correctly that Sami's claim was incorrect. They accepted the equivalent definition of the parallelogram as an accurate definition. For example, Shahaf explained, "I think that Sami's claim is wrong because 'diagonals cross each other' is indeed an attribute of the parallelogram and it can be said that the above sentence can serve us as proof that the quadrilateral is a parallelogram and therefore it can also be used as a definition". Samer's explanation was, "It is true that this is an attribute, but it can be used as a definition, since the quadrilateral meets it if and only if it is a parallelogram". Shahaf and Samer know that when we use necessary and sufficient attributes that characterize a geometric concept they can serve us as a formal definition.

Table 4. Does the criterion of crossing diagonals classify a parallelogram?

	Sami's claim is correct	Sami's claim is incorrect	Other	Total
The criterion classifies parallelograms	21 39.62%	14 26.42%	2 3.77%	37 69.8%
The criterion does not classify parallelograms	9 16.98%	6 11.32%	1 1.89%	16 30.19%
Total	30 56.6%	20 37.74%	3 5.66%	53 100%

As seen in Table 4, the great majority (about 70%) of the participants claim correctly that the attributes of crossing diagonals in quadrilaterals could be a criterion for classifying examples and non-examples of the parallelogram concept. Out of those participants, more than half (57%) reject the idea that the same attribute could be used as a definition of the parallelogram concept (i.e., Sami's claim is right). Again, this fails to meet van Hiele & van Hiele's (1958) fourth level. Only about 38% of the participants who accept that quadrilaterals with crossing diagonals could be a criterion accept using the same attribute as a definition (Sami's claim is incorrect).

Interview 2 (Reem)

Interviewer: Can we use the criterion "quadrilateral in which the diagonals cross each other" to sort examples and non-examples of the parallelogram concept?
Reem: Yes, we can.
I: Rami defined the parallelogram as "a quadrilateral in which the diagonals cross each other". Is it a correct definition?
R: No, it is not a definition; the definition is that in a quadrilateral every opposite sides are parallel. What you said is a characteristic of the concept. The definition that all of us agree with is "a quadrilateral in which every two opposite sides are parallel".
I: Can we prove that a quadrilateral whose diagonals cross each other is a parallelogram?
R: Yes, we can and it is easy to prove it.
I: You write that the definition's role is to identify examples of the geometric concept.
R: Yes.
I: Is [said attribute] appropriate for identifying and classifying examples and non-examples of the parallelogram concept?
R: Yes.
I: Could it be a formal definition of the parallelogram concept?
R: I think yes. Before, I learned that for every concept, there is only one definition and the rest are the attributes that we can derive from the definition. Now I am learning something new.
I: What have you learned?
R: For geometric concept might could be more than one exclusive definition.

Reem's position regarding the possibility of more than one definition of a particular geometric concept changed for two reasons. At first, Reem understood that when necessary conditions were sufficient to sort examples and non-examples of a particular geometric concept, and then it could serve us as a mathematical definition of the concept. Next, Reem understood that we could switch from one definition to another by the way of proof.

Parallelogram Definition (Tasks 5&6)

Recall that Definition 1 for a parallelogram was "quadrilateral in which every two opposite sides are parallel"; Definition 2 was: "quadrilateral in which every two opposite sides are equal".

As shown in Table 5, many participants (40%) were incorrect in their claiming that only definition 1 was correct (50% of the senior teachers, 31% of the pre-service senior teachers and 40% of the pre-service junior teachers). Comparing the three subsets, surprisingly enough that, more senior teachers answered incorrectly that only the first definitions was correct. Tamir, Udi, Nofar and Silve's explanation was, "The first is a definition and the second is an attribute derived from the definition". Nevertheless, all of them accepted that “quadrilateral in which every two opposite sides are equal" was sufficient for the shape to be a parallelogram. Yossi claimed that only the first definition was correct, consistent with his explanation about the essence of the concept: "If you refer to the definition as it is, then only the first definition is correct, because it appears as far as I remember in the textbooks. The second […] is not a definition but an attribute of a parallelogram. It is important to note that the definition is not arbitrary and cannot be replaced by other attributes simply because the definition tries to grasp the core essence of the geometrical form". I assume that the textbooks influence the senior teachers' responses; I see this clearly in Yossi's response. The textbooks in senior schools emphasize that each concept has one exclusive definition and from it the other attributes derived.

Table 5. Two definitions of parallelogram

	Only Definition 1 is correct	Only Definition 2 is correct	Both definitions are correct	Other / Both definitions combined are correct	Total
Preservice junior math teachers	9 16.98%	3 5.66%	7 13.21%	3 5.66%	22 41.51%
Preservice senior math teachers	6 11.32%	0	12 22.64%	1 1.89%	19 35.85%
Inservice Senior math teachers	6 11.32%	0	4 7.55%	2 3.77%	12 22.64%
Total	21 39.62%	3 5.66%	23 43.4%	6 11.32%	53 100%

Only 43% of all participants (33% of senior teachers, 63% of pre-service senior and about 32% of pre-service junior teachers) claimed correctly that the both definitions were correct. Maile, Shahaf, Ayal and Samer accepted that "quadrilateral in which every two opposite sides are equal" was a sufficient attribute for a quadrilateral to be a parallelogram and that both definitions were correct. Ayal gave “the same weight" to both definitions and explained: "The geometrical shape of parallelogram corresponds to the two definitions. Both can be proved and both can be used to solve problems". However Mile, Shahaf and Samer claimed that both definitions were correct, they gave "more weight" to Definition 1. Samer explained: "The definition of the parallelogram concept is the first, while the second definition is the equivalent definition". Mile explained: "Definition 1 is more accurate; from Definition 2 we can infer that if all two pairs of opposite sides are equal, they are necessarily also parallel". Shahaf explained her answer as follows: "Basically both definitions are suitable for the parallelogram concept. According to what I know, the formal definition of a parallelogram is "a quadrilateral with two pairs of parallel opposite sides" or "a quadrilateral in which two opposite sides are parallel", but once a quadrilateral of two pairs of opposite sides are equal then it can be defined as a parallelogram".

About 11% of participants claimed that the correct definition would be a combination of both. Nadav explained: "Each definition separately describes some of the parallelogram properties together to help define the parallelogram concept".

Table 6 presents an analysis of Tasks 5 and 6. It shows that about 83% of participants claimed correctly that the attribute, quadrilateral in which every two opposite sides are equal, could be a criterion for classifying examples and non-examples of the parallelogram concept. Only 50% of them claimed correctly that both definitions were correct. Another third of those participants rejected the use of the same attribute as a definition of the parallelogram concept. They answered that only definition 1 was correct. In addition, 14% of them demanded another attribute (such as put the two definitions together), for the definition of a parallelogram to be appropriate and formal. Nofar, for example, claimed that the attribute "every two opposite sides are equal" was an acceptable classification criterion, but did not accept the same attribute as a definition for the parallelogram concept.

Table 6. Does the criterion that every two opposite sides are equal classify a parallelogram?

	Only Definition 1 is correct	Only Definition 2 is correct	Both definitions are correct	Other / Both definitions combined are correct	Total
The criterion classifies parallelograms	14 26.4%	2 3.77%	22 41.51%	6 11.32%	44 83.02%
The criterion does not classify parallelograms	7 13.21%	1 1.89%	1 1.89%	0	9 16.89%
Total	21 39.62%	3 5.66%	23 43.4%	6 11.32%	53 100%

Interview 3 (Nofar)

Interviewer: You claim that the criterion "quadrilateral in which every two opposite sides are equal" could be used to classify examples and non-examples of parallelogram concept.
Nofar: Yes I do.
I: And you claim that a quadrilateral in which every two opposite sides are equal could not be a definition for the parallelogram concept?
N: Yes, I do.
I: Why?
N: Because there is one definition: a quadrilateral in which every two opposite sides are parallel called a parallelogram. This definition clarifies the meaning of the name and the essence of the concept.
I: And what about a quadrilateral in which every two opposite sides are equal?
N: This does not give us the same meaning.
I: One student asked to prove that a certain shape is a parallelogram; he proved that a quadrilateral in which every two opposite sides are equal [was a parallelogram]. Is this proof sufficient and correct?
N: Yes. Because it is an attribute that is derived from the definition of parallelogram. In addition, from a quadrilateral in which every two opposite sides are equal we can infer that every two opposite sides are parallel.
I: And you are sure about your answers before?
N: Yes and the uniformity is important.

Although Nofar claims that, the role of the definition is to classify examples and non-examples of the concept and accepts that a quadrilateral in which every two opposite sides are equal could be a classification criterion, but like Yossi it is important for her that the definition highlights the meaning and the essence of the geometrical concept. Nofar understands that from the attribute "quadrilateral in which every two opposite sides are equal" we can infer "quadrilateral in which every two opposite sides are parallel" and vice versa, but still thinks that the attribute quadrilateral in which every two opposite sides are equal could not be a definition of the parallelogram concept. The interviews findings with Yossi, Reem and Nofar reinforce the emerging trend from the questionnaire findings. The understanding that attributes could be classification criteria does not help participants overcome their difficulty or misconception in understanding that for particular geometric shape might be more than one equivalent and economical definitions.

Discussion

The aim of the study is to investigate whether understanding the role of definitions as classification and identification criteria will guarantee that participants: (1) accept that there may be more than one equivalent definition for particular concept; and (2) accept all minimal definitions which include necessary and sufficient attributes to be legal definitions. I can see a clear trend whereby the identification that certain attributes are necessary and sufficient to be a classification criterion of concept examples and non-examples does not guarantee that the participants think about the same attributes as an equivalent or minimal mathematical definition for the same concept. The majority (57%) among the participant who accept that quadrilaterals with crossing diagonals could be a criterion for classifying the parallelogram concept do not accept using the same attribute as an equivalent definition for the parallelograms (see Table 4). About one third (32%) of participants who accept that the attribute quadrilateral in which every two opposite sides are equal could be criteria for classifying examples and non-examples of the parallelogram concept did not accept using the same attribute as an equivalent definition for parallelogram (see Table 6). Finally, 25% of participants who claim correctly that the attribute two triangles, which have two congruent angles as a classification criterion for similar triangles did not accept it as the minimal and equivalent definition for similar triangles (see Table 2). Perhaps using the similar triangles theorem (A, A) helps the majority of participants to accept that two triangles, which have two congruent angles as a legal definition for similar triangles. However, for many participants this is insufficient as a minimal definition and they consider it as a theorem. There is a distinction between definitions, theorems and proofs. What the teachers need to realize is that the similar triangles theorems emphasize the minimal definition of the similar triangles concept.

My findings regarding the difficulties of pre-service, but also in-service teachers in understanding the geometric definitions confirm those of previous studies. They indicate two major difficulties: the difficulty to identify economical, parsimonious or minimal definitions (Linchevsky, Vinner & Karesnty, 1992; de Villiers, 1998) as in the similar triangles task, and the inability to identify or find equivalent definitions (Harel, Selden & Selden, 2006; Haj Yahya, Hershkowitz & Dreyfus, 2014; Leikin & Winicky-Landman, 2001) as in the parallelogram tasks. The influence of the difficulty to identify equivalent definitions in accepting a geometric definition to be a legal one was found to be stronger than that of the difficulty to identify economical definitions: 36% of participants did not accept an economical definition and about 56% did not accept equivalent definitions. This result coincides with the findings of Haj Yahya, Hershkowitz & Dreyfus (2014).

Two reasons may be responsible for these difficulties. The first reason is the failure to understand the logical structure of concept properties and failure to infer one property of the concept from others. This ability is expected at van Hiele & van Hiele's (1958) third level. The second reason is that many participants think that for mathematical concepts there is only one exclusive definition, while all other statements are attributes, so that uniformity must maintained within the mathematician and mathematical education communities (Borasi, 1992; Leikin & Winicky-Landman, 2001). I suppose likelihood of the second reason is greater because the participants deal with the concerning concepts in the study in their daily life when learning mathematics in the college or teaching mathematics in the senior schools.

For the majority of the participants the definition is a description (see Table 1) therefore they prefer a definition, which describes the essence of the concept: they prefer a descriptive to creative definition (Freudenthal, 1973). I can see that it is very important for the participants that the definition include the attributes that highlights the meaning and the essence of the concept. Thus, for example, many participants in the questionnaire prefer to outline the superfluous attribute of proportional sides in similar triangles and are not satisfied with the equality of angles. Another evidence for this I can see in the interview with Yossi. Yossi accepted the definition for similar triangles when all their corresponding sides have lengths in the same ratio rather than, when two angles of one triangle have the same measure as two angles of another triangle because the first highlight the essence of the concept and therefore it can constitute a formal definition for similar triangles.

We can argue about what a good definition is, but we have to agree that when attributes are necessary and sufficient for classifying the examples and non-examples of a concept, then they can constitute a formal definition. For many participants, the essence of the mathematical concept (Mariotti & Fischbein, 1997) is more important than the essence of the mathematical definition (Leikin & Winicky-Landman, 2001). From a pedagogic point of view, one should not adhere to minimal definition in the case of similar triangles because the non-minimal definition emphasizes the essence and the meaning of the concept (Zaslavsky & Shir, 2005), but the students must understand that the minimal definition correct and valid definition, this is the emphasis on mathematics as a logical science.

The deductive level is the fourth in van Hiele & van Hiele's levels (1958). The abilities expected at this level include understanding the meaning of necessary and sufficient conditions as characteristics of formal definitions and understanding the equivalence of concepts definitions. Even those who accept an equivalent definition as a formal definition give more "weight" to one of the definitions, without realizing that if one definition may be inferred from the other, the both definitions are equivalent and have the "same weight". These participants operate according to what expected at van Hiele and van Hiele's (1958) third level: they understand the importance of the accurate definition, they identify a minimum set of attributes that can characterize a shape, but they do not understand what a definition is and what role it plays in the deductive structure of geometry.

Conclusions

The present research intended to examine the effects of mathematics pre-service and in-service teachers' perception of geometric concepts definitions on their defining process. I can see a general and clear tendency:

teachers' and pre-service teachers' difficulties in perception the characteristics and the roles of mathematical definitions of geometrical concepts affect these participants' defining processes. In this study, I can see a fact of existence; although a particular attribute or subset of attributes can serve as a definition of the geometric concept and have the characteristics of the mathematical definition; many participants rejected it from being a legal definition of the same concept. To say more explicitly: understanding that an attributes or a set of attributes could be a classification and identification criteria because it include necessary and sufficient attributes; did not lead that the participants understand that the same attributes could be a geometrical definitions and therefore they rejected equivalent and minimal definitions as legal definitions. The research results help plan educational researches that targets the features of teachers' conceptions of definitions. These researches would constitute a platform for discussing the impact of mathematics teachers' conceptions of definitions on their teaching.

Limitations, Future Directions and Practical Implications

The main limitation of this study is the use of a small convenience sample. The total number of participants in the study was relatively small, and the three subgroups were even smaller, preventing me from examining significant differences between them (two groups were smaller than 20 participants). Moreover, the fact that it did not use larger and more representative research population prevents from generalizing to the teacher population. Therefore, it is important to carry out further studies in the future in order using a larger and more diverse research population. In particular, future studies should also include high school students, as well as populations from different sectors in society and other parts of the world. This would allow us to determine whether cultural differences, for example, affects the findings. It would also be interesting to use of different methodologies, such as classroom viewing to gather more qualitative information about the population under study: it would be very interesting to see what emerges within the classroom discourse during such lessons in order to learn about the thinking processes of both teachers and students, and most importantly, their interaction.
To conclude, my recommendation for geometry instruction in the area of teacher training. It is better that teachers be exposed during their training to the specific difficulties that have been revealed in this study. This will raise their awareness of the processes that lead to these difficulties and sensitize them to coping with them in the teaching process. Creating such a mindset and motivation might help mathematics teachers to diagnose and think through students' difficulties, perform better as teachers, and improve students' achievements.

Notes

This article is dedicated to the memory of Muhammad Haj-Yahya, my colleague in the mathematics teachers' staff in Amal College in Taybee.

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Appendix A: The Questionnaire Form

1. Can we use the criterion "two angles of one triangle are identical to two angles of another triangle" to sort examples and non-examples (counter examples) of two similar triangles?
   a) Yes we can.*
   b) No we cannot
   c) Another answer…
2. Two students debated about defining similar triangles. Sami said, "two triangles, $\triangle ABC$ and $\triangle A'B'C'$, are similar if and only if corresponding angles have the same measure and the lengths of corresponding sides are proportional". Rami said that there is a superfluous condition in Sami's definition and suggested the following definition: "two triangles, $\triangle ABC$ and $\triangle A'B'C'$, are similar if they have two congruent angles".
   Which definition/s is/are correct? Explain your answer!
   a) Only Sami's definition is correct.
   b) Only Rami's definition is correct.
   c) The two definitions are correct but I prefer Rami's.*
   d) The two definitions are correct but I prefer Sami's.*
   e) Both definitions are incorrect.
   f) Another answer…
3. Can we use the criterion "a quadrilateral in which the diagonals cross each other" to sort examples and non-examples (counter examples) of the parallelogram concept?
   a) Yes we can.*
   b) No we cannot
   c) Another answer…
4. Rami defined the parallelogram as "a quadrilateral in which the diagonals cross each other." Sami claimed that Rami's definition was incorrect and noted that Rami used an attribute of the parallelogram derived from the definition that "a parallelogram is a quadrilateral in which every two opposite sides are parallel". Is Sami's claim correct? Explain your answer!
   a) Sami's claim is correct.
   b) Sami's claim is incorrect.*
   c) Another answer…
5. Can we use the criterion "quadrilateral in which every two opposite sides are equal" to sort examples and non-examples (counter examples) of the parallelogram concept?
   a) Yes we can.*
   b) No we cannot
   c) Another answer…
6. Read the following definitions of the parallelogram concept. Definition 1: "A parallelogram is a quadrilateral in which every two opposite sides are parallel". Definition 2 "A parallelogram is a quadrilateral in which every two opposite sides are equal".
   Which definition is correct? Explain your answer!
   a) Only the first definition is correct.
   b) Only the second definition is correct.
   c) Both definitions are correct.*
   d) Neither definition is correct.

*Correct answers