Understanding Inclusion Relations between Quadrilaterals

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Article Info	Abstract
	This study investigated pre-service mathematics teachers' (PST), who are going to teach middle grade mathematics (grade 5-8), understanding of inclusion relationships of quadrilaterals. In addition to describing PSTs' understanding of inclusion relations of quadrilaterals; it was also aimed to document the development of PSTs' understanding of quadrilaterals during a geometry course. A designed questionnaire was administered to 48 PSTs at the beginning and again at the end of the semester after the PSTs was engaged in a designed five-week geometry unit. The findings of this study demonstrated that the majority of the PSTs struggled with identifying quadrilaterals and especially inclusion relations of quadrilaterals primarily. However, the number of the PSTs who understood hierarchical relationship between quadrilaterals increased through the end of the semester.

Introduction

Researchers argue that learning to identify geometric shapes and to understand hierarchical classification among these shapes is prerequisite for learning more complex concepts such as spatial reasoning or deductive thinking (e.g. Clements, 2003; Fujita & Jones, 2007). Hierarchical classification means the classification of a set of concepts in such a manner that more particular concepts form subsets of the more general concepts (de Villiers, 1994). Despite the importance of understanding geometric shapes and hierarchical classification among quadrilaterals, Fujita and Jones (2006) claim that pre-service teachers' subject knowledge of geometry is amongst their weakest knowledge of mathematics. Previous studies support Fujita and Jones' claim by documenting pre-service teachers' difficulties with quadrilaterals, especially with inclusion relations of quadrilaterals (e.g, Fujita & Jones, 2006, 2007; Fujita, 2012; Türnüklü, 2014; Zilkova, 2015).

The number of studies that address teachers and pre-service teachers' understanding of inclusion relations of quadrilaterals is limited or solely focus on pre-service elementary teachers, whose subject matter knowledge of mathematics has been the center of major concerns. Furthermore, these studies explore the process of pre-service teachers' understanding of inclusion relations of quadrilaterals and/ or the cognitive path (Fujita, 2012; Türnüklü, 2014). However, these studies only focus on investigating relations between some special quadrilaterals (i.e. parallelograms) while excluding other quadrilaterals such as trapezoids or kites.

The purpose of this study is twofold. First, it is to document pre-service mathematics teachers' (PSTs') understanding of inclusive relations of all special quadrilaterals including trapezoids and kites. Then, it is to document how a designed geometry unit might help PSTs' develop their understanding of quadrilaterals. The following two research questions guided this study:

1- What do pre-service teachers' know about the inclusion relations of quadrilaterals?
2- How PSTs' understanding of inclusion relations of quadrilaterals has changed during a geometry course?

Literature Review

Thinking process of students in geometric contexts has sparked the interests of many researchers (e.g. Fujita, 2012; Fujita & Jones, 2007; Türnüklü, 2014; van Hiele, 1986). There are various theories dealing with the

development of geometric thinking and learning of children as well as adults. In this section theories of geometric thinking and learning will be addressed.

Overview of the van Hiele Model

According to the theory of Pierre and Dina van Hiele, students progress through hierarchical levels of thought in geometry (Clements, 2003; van Hiele, 1999). The van Hiele model of geometric thought consists of five sequential levels labeled as: "visualization," "analysis," "informal deduction," "formal deduction," and "rigor" (van Hiele, 1986). Students usually achieve levels from visualization to analysis in early elementary grades while they can reach to informal deduction level either in upper elementary or middle grades and formal deduction in secondary grades or in college (Clements & Sarama, 2000).

At visualization level, geometric concepts are viewed as a whole rather than as having components or attributes. Students can recognize and identify two-dimensional shapes by their visual appearances (Clements, 2003; van Hiele, 1999). At analysis level, on the other hand, an analysis of geometric concepts and their basic properties begins. Students recognize the properties of two-dimensional shapes and they understand that all shapes within a class share common properties (van Hiele, 1999). When students reach at informal deduction level, they can establish the interrelationships of properties among shapes. Class inclusion is understood at this level. Students still, however, do not grasp that logical deduction is the method for establishing geometric truth (Clements, 2003). At level 4, formal deduction, students can establish theorems within axiomatic system. Rigor is the level where students investigate shapes in other axiomatic systems such as spherical geometry (van Hiele, 1986, 1999). Existing research supports that the van Hiele levels are useful in describing students' geometric concept development (Burger & Shaughnessy, 1986; Clements & Battista, 1992). However, these levels are also criticized for being too broad (Clements, 2003). Thus, several researchers have recently offered variations of the original five van Hiele levels (e.g. Clements & Battista, 1992; Lehrer, Jenkins, & Osana, 1998), which will be described next.

Classification of Understanding of Inclusion Relations of Quadrilaterals

To document learners' understanding of quadrilateral classifications, especially understanding inclusion relations of quadrilaterals, "concept image" as proposed by Tall and Vinner (1981) and "figural concept" for geometric shapes by Fischbein (1993) have an essential role as the theoretical background. Concept image is used to define the total cognitive structure that is associated with the concept, which includes all the mental pictures (i.e. pictorial, symbolic or graphical representations) and associated properties and processes (Tall & Vinner, 1981, 2002; Vinner, 1983; Vinner & Dreyfus, 1989; Vinner & Hershkowitz, 1980). The definition of a concept, on the other hand, is a form of words used to specify that concept (Tall & Vinner, 1981; Vinner, 1983). Studies document that students often use concept images rather than definitions of concepts in their reasoning (e.g. Clements, 2003; Vinner & Dreyfus, 1989). Fischbein (1993) used the term figural to refer to the dual nature of geometrical concepts that are understood both through perceptions of examples and through logical features. While Vinner and Hershkowitz's (1980) concept image links the associated mental images and properties of a mathematical concept, Fischbein's (1993) figural concept is unique to geometric concepts (Walcott, Mohr, & Kastberg, 2009).

Table 1. Levels of understanding of inclusion relations of quadrilaterals

Levels	Descriptions
Level 3: Hierarchical	Learners can understand hierarchical relationships among quadrilaterals. The inclusion relationships of quadrilaterals are understood and can be used for all quadrilaterals. 'The opposing direction inclusion relationship' of definitions and attributes is understood.
Level 2: Partial Prototypical	Learners have begun to extend their figural concepts to understand inclusion relations of quadrilaterals. However, their understanding is limited and specific to some quadrilaterals.
Level 1: Prototypical	Learners have their own limited personal figural concepts. Their judgments regarding to identifying relationships of quadrilaterals are judged by their limited figural concepts.
Level 0	Learners do not have basic knowledge of quadrilaterals.

Fujita (2012) proposes a theoretical model to describe cognitive development of understanding of inclusion relations of quadrilaterals by synthesizing past theories described briefly above (i.e. concept image, figural concepts or van Hiele theory). However, Fujita (2012) explicitly focused on the relationships between certain special quadrilaterals—parallelograms with squares, rectangles, and rhombi and the relationship between squares with rectangles. Their classification levels of these specific quadrilaterals were adopted; but extended to include other quadrilaterals such as trapezoids or kite. These classification levels were used to analyze the responses of the participants.

Method

Participants

48 PSTs, who are going to teach middle grade mathematics (grade 5-8) upon graduation, participated in this study. Participants enrolled in a three-credit geometry course, which was taught by the author when this study was conducted, at a large public university in Turkey. It is a prerequisite course that all the PSTs have to take in their first year of the teacher education program at the university. The participants were freshman when the study was conducted and all of them took high school geometry prior to this study. The geometry course incorporated a designed quadrilateral unit for five weeks. The participants were chosen by purposeful sampling method used for qualitative research studies (Patton, 1990).

Data Sources

The data sources, which were used to address the research questions, were as follows:

(1) A geometry questionnaire with sixteen open-ended questions was designed to investigate PSTs' subject matter knowledge of geometry. The questionnaire was administered to whole class in the first and last week of the semester in the spring of 2016. The PSTs were provided 1 hour to answer all of the questions in the questionnaire individually. The questions in the questionnaire were designed to check participants' figural concepts of quadrilaterals, to check whether they can use different names to define quadrilaterals and how they understand the relations between quadrilaterals, as well as to judge whether PSTs have a static or rather dynamic images of quadrilaterals.

(2) A quadrilateral unit, which consisted of five weeks of lesson plans, was designed to help PSTs develop their knowledge of quadrilaterals, especially inclusion relations of quadrilaterals. During the implementation of this quadrilateral unit, the PSTs were engaged in several activities and asked to work in groups of 4-6 during the course. A stable camera was located at the back of the class to capture the board and videotaped all instructions.

The purpose of the first week's lesson plan was to explore various examples of seven special quadrilaterals including square, rectangle, parallelogram, rhombus, trapezoid, isosceles trapezoid, and kites and their side, angle, diagonal and symmetry properties (see Appendix A for a sample). The purpose of the lesson for the second week was to define special quadrilaterals and explore various definitions as well as definition types (hierarchical versus partial definitions) (see Appendix B for a sample). The lesson for Week 3 aimed to identify relationships among special quadrilaterals while the lesson for Week 4 aimed to use these inclusion relations to construct various quadrilaterals by using only compass and straightedge. The last week of the geometry unit provided the PSTs information about one of the mostly known models to explain geometric thinking, van Hiele theory, and how this model could be used to design/select tasks for teaching geometry in the middle grades. Existing literature guided the process of planning each lesson in the geometry unit. More details about each week's lesson plan will be addressed in the results section.

(3) The PSTs' midterms, class works, and assignments were also collected during the semester.

Data Analysis

The framework adopted from Fujita (2012) guided the analysis of the data (see Table 1). PSTs' responses to the questionnaire items and their class work and assignments were coded in three maincategories (Level 0 was excluded since all the participants have taken high school geometry and have basic knowledge of quadrilaterals). However, for this study Level 1 and Level 2 were further divided into subcategories based on the

PSTs' responses. Although the levels proposed by Fujita (2012) provide such an essential model to describe learners' understanding of inclusion relations of quadrilaterals, these levels seem to be too broad. The responses of the PSTs to the questionnaire items differed in terms of their understanding of inclusion relations. For instance, It was observed that while some PSTs only extended their figural concepts of a couple of quadrilaterals (i.e. seeing square as a subset of rhombus or rhombus as a subset of parallelogram), some PSTs were able to understand the inclusion relations between almost all of the quadrilaterals (except seeing special quadrilaterals with at least one pair of parallel sides as subsets of trapezoids). Categorizing all these PSTs in the same level—Level 2— seemed to be too broad and not capturing the PSTs' levels of thinking properly. Therefore, Level 2—Partial Prototypical Reasoning—was further sub-divided. The sub-levels of the framework will be described further in details in the discussion section.

Data analysis began by examining the written work of each participant and grounding it in a constant comparative method of coding (Charmaz, 2006; Glaser & Strauss, 1967) in which participant responses were coded with external and internal codes. Coding of the data began with a set of external codes—Prototypical Reasoning, Partial Prototypical Reasoning and Hierarchical Reasoning—that were derived from the framework proposed by Fujita (2012) (see Table 1). Such external coding schemes provided a lens with which to examine the data. However, examining the data and reviewing the written responses of the participants in each category further, internal codes were also developed to capture inclusion and transitive relations of quadrilaterals (see Table 2). After proposing these internal, data-grounded codes—Level 1-a, Level 1-b, Level 2-a, Level 2-b, Level 2-c— each written work was reexamined and recorded to incorporate these new codes.

Results

The PSTs' responses to the questionnaire items are displayed cumulatively in Table 2 below. As can be seen in the table, participants demonstrated mainly partial prototypical reasoning—38— PSTs at the beginning of the semester. Among the PSTs who were categorized in Level 2, 19 PSTs were coded in Level 2-c—Partial prototypical reasoning without transitive understanding. That is, the PSTs were able to reason that a class of quadrilateral was a subset of another class of quadrilateral, but they were not able to understand that if quadrilateral A was a subset of quadrilateral B which was a subset of quadrilateral C then quadrilateral A was also a subset of quadrilateral C.

Table 2. Pre and post questionnaire results

Levels		Pre-Questionnaire (N=48)	Post-Questionnaire (N=48)
Level 3: Hierarchical		-	29
Level 2: Partial Prototypical	Level 2-a: Hierarchical Reasoning without Opposing Direction Inclusion Relations	11	18
	Level 2-b: Partial Prototypical Reasoning with Transitive Understanding	8	1
	Level 2-c: Partial Prototypical Reasoning without Transitive Understanding	19	-
Level 1: Prototypical	Level 1-a: Exceptional Prototypical Reasoning	4	-
	Level 1-b: Prototypical Reasoning	6	-

Given that all participants had taken high school geometry course, it was not surprising to find that the PSTs held inclusive reasoning regarding to at least some quadrilaterals. However, it was surprising to find that none of the participants demonstrated hierarchical thinking at the beginning of the semester. Even though the PSTs struggled with identifying inclusion relations of quadrilaterals, the majority of them were able to understand inclusion relations of quadrilaterals by the end of the semester. However, it should be noted here that 18 PSTs still struggled to understand opposing direction inclusion relations by the end of the semester. Participants' responses to the questionnaire questions at the beginning and the end of the semester will be analyzed in details next.

Pre-Questionnaire Results

The analysis of the PSTs' responses to the questionnaire items in the first week of the semester demonstrated that the participants struggled in identifying inclusion relations of quadrilaterals. As can be seen in Table 2 above, the majority of the PSTs (38 out of 48 PSTs) demonstrated partial prototypical reasoning. That is, they demonstrated some sorts of difficulty understanding inclusion relations of quadrilaterals. While 38 PSTs demonstrated partial prototypical reasoning, 10 PSTs demonstrated prototypical reasoning. Among these 10 PSTs, 6 PSTs held entirely prototypical reasoning about quadrilaterals. Four PSTs, on the other hand, were able to extend their figural concept of one specific quadrilateral, which was either recognizing a square as a rhombus (3 PSTs out of 4) or a rhombus as a parallelogram (1 PST). Since these four PSTs relied heavily on their prototypical examples for all quadrilaterals, except just for one specific quadrilateral, they were coded in Level 1. However, these PSTs were acknowledged as in transition to Level 2-c.

Investigating the responses of the PSTs to the questionnaire items revealed that the majority of the PSTs struggled to identify trapezoids the most. The PSTs either held incorrect conceptions of what constitute a trapezoid or they had prototypical view of trapezoids. For instance in the following response, The PST selected the shapes E, F, J, N, P, S as trapezoids as opposed to including the shapes A, C, D, F, H, K, L, O, and T (the shapes with at least one pair of parallel sides). Then, the PST explained that the shapes were four sided but the sides were not regular. It was evident in his response that the PST knew that a trapezoid was a quadrilateral, therefore should have four sides. However, he might not be clear about what constituted a side in polygons since he listed shape N as a trapezoid. Additionally, it was evident in his response that he was not aware of the fact that in order for a quadrilateral to be considered as a trapezoid, it should have at least one pair of parallel sides. Instead, the PST listed the shapes E, J, P and S as trapezoids even if it was clear that they did not have any parallel sides.
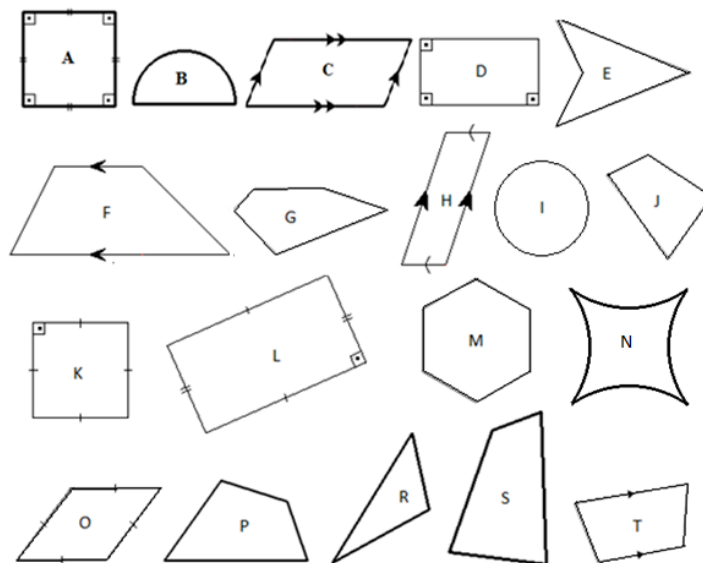
3- List all the shapes that you think is a **Trapezoid**.
   Answer: E,F,J,N,P,S
   Reason: 4 sided and sides are not drawn regularly

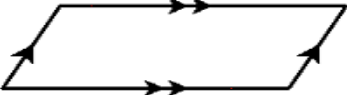Figure 1. A PST's response to what a trapezoid is

Similarly, one of the PSTs, as displayed in Figure 2, was able to recognize inclusion relationships between all quadrilaterals but trapezoids. It was apparent in the PST's response that he only selected the name of trapezoids for the shapes 5 and 6, which resembled the prototypical examples of trapezoids the most. Although he selected the names correctly for other quadrilaterals, he crossed off the name of trapezoids for all quadrilaterals other than 5 and 6.It was evident in his response that he did not see square, rectangle, parallelogram or rhombus as examples of trapezoids. By selecting shape 5 as an example of a trapezoid, it seemed like he did not know the fact that a shape should have at least a pair of parallel sides in order to be considered as a trapezoid either. In this item, the participants were expected to select the name of trapezoid for shapes 2,3,4,6, and 7 and crossed off the name of trapezoid for the shape 5.

Figure 2. A sample response to a questionnaire item

In this study, the PSTs at Level 2-c were able to recognize the inclusive relationships between a few quadrilaterals. When investigated their responses further in details, it was apparent that they usually were able to recognize the interrelationships between a square with a rhombus or a rhombus with a parallelogram. Okazaki (1995) suggests that some inclusion relations between quadrilaterals are easier to grasp for learners than others. It was evident in the findings of this study that square - rhombus and then, rhombus - parallelogram associations were easier for the PSTs to draw. The most difficult association for the participants, on the other hand, was to establish between all quadrilaterals with at least one set of parallel sides such as parallelogram, rhombus, square, rectangle with trapezoids.

Furthermore, when the PSTs' responses were investigated further, it was clear that transitive relations were missing in the participants' reasoning coded in Level 2-c. For instance as can be seen in one of the PSTs'

response below (Figure 3), the PST was able to state that a square was also a rhombus since all sides were equal in squares as they were in rhombi. Thus, the PST was able to see squares as a sub-set of rhombi given that the critical attribute of rhombi also applied to squares. Similarly, the PST was able to state that a rhombus was also a parallelogram since opposite sides were parallel and equal in both. However, he failed to realize that squares then form a subset of parallelograms. Erez and Yerushalmy (2006) argue that understanding the transitive relations between quadrilaterals is essential to understand hierarchical relationships of quadrilaterals. Since the PSTs who were coded at Level 2-c did not yet comprehend this essential understanding for hierarchical reasoning, they did not yet reach hierarchical reasoning level.
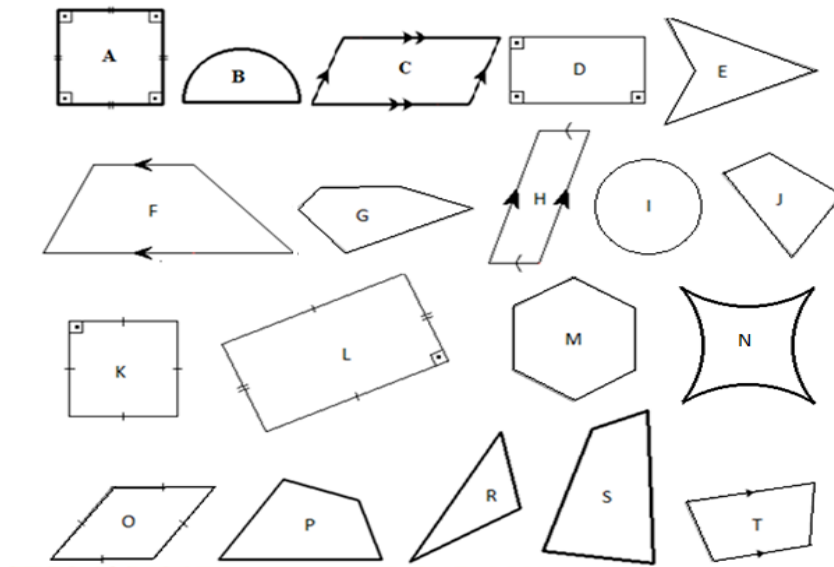
2- List all the shapes that you think is a **Rhombus.**
   Answer: A , K, O
   Reason: Becasue all sides are equal

5- List all the shapes that you think is a **Parallelogram.**
   Answer: C, H, O
   Reason: Because opposite sides are parallel and equal

Figure 3. A PST's response the questionnaire item 1 in the pre-questionnaire

Even though this transitive reasoning was missing among the PSTs who were coded at Level 2-c, the PSTs, who were coded at Level 2-b, were able to understand transitive relations between quadrilaterals. For instance, they were able to recognize that that if a square was a rhombus and a rhombus was a parallelogram then a square was also a parallelogram.

Finally, the PSTs' responses to the questionnaire items showed that they primarily held inflexible mental image of quadrilaterals, which resulted in identifying inclusion relations of quadrilaterals only in one way. The PSTs, who were coded in Level 2-a, were able to recognize that a rectangle was a family covering squares— each square had all rectangle properties. However, the PSTs did not comprehend the situation of opposing direction inclusion relations (Erez & Yerushalmy, 2006; Okazaki & Fujita, 2007), which resulted in claiming that a rectangle could not have square properties. Wilson (1990) refers to fixed visual prototypes as prototypes that are set and not easily manipulated within the learner's mind, as inflexible. The PSTs in Level 2-a were not able to envision movement upon the shapes. Rather, they held inflexible mental images of the shapes and failed to grasp the dynamic behavior, which could preserve the critical attributes of the shapes. In the following response, the PST argued that a rectangle was always a parallelogram, but parallelogram was never a rectangle. Even though she knew the critical attributes of all the quadrilaterals and their hierarchical relations, but she did have a flexible mental images of the shapes, which prevented her from coordinating the mental models of quadrilaterals with the understanding of mathematical meaning.

8– Fill in the blanks with "**Always**", "**Sometimes**", or "**Never**".

a) A square is........................*never*........................ a trapezoid.

b) A rhombus is........................*always*........................a parallelogram.

c) A parallelogram is........................*never*........................a rectangle.

d) A trapezoid is........................*never*........................a parallelogram.

e) A rectangle is........................*never*........................a square.

f) A square is........................*always*........................a rectangle.

g) A rectangle is........................*never*........................a rhombus.

Figure 4. A sample response to a questionnaire item in the pre-questionnaire

Quadrilateral Unit

Geometry is an essential part of mathematics curriculum (NCTM, 2000). During 5 weeks, the PSTs were engaged in several activities in order to define quadrilaterals, investigate their properties and realize the inclusion relationships among them. In this section, the activities that the PSTs were engaged during the quadrilateral unit will be described in details.

Week 1

Week 1 activities were designed to help PSTs develop and recognize the variety of the quadrilaterals and explore their properties. The PSTs were provided geo-boards, dotted paper, and polystrips to work with during the class. They were first asked to make (draw) quadrilaterals one by one on their geo-boards, dotted papers or by using their polystrips. Then, they were asked to investigate side, diagonal, angle, and symmetry properties and make a list of properties for each quadrilateral one by one in their groups (see Appendix A for a sample activity sheet). Watson and Mason introduce the idea of example spaces, which are collections of examples that fulfill a specific function (Watson & Mason, 2005). Walcott et al. (2009) argue that the classification of geometric shapes is often based upon comparisons to a visual prototype or exemplar that represents a particular class. In order to help PSTs develop their examples spaces and recognize non-prototypical shapes as example of a particular class, they were encouraged to construct various examples of each quadrilateral during exploring their properties. For instance, during exploring the properties of kites, the examples below were discussed— whether they both belong to a kite family.

Figure 5. Examples of kites used during instruction

Week 2

In week two, the PSTs were asked to identify the similarities and differences between each quadrilateral in their groups first and then to discuss as a whole class. During the whole class discussion, the PSTs were encouraged to discuss about distinguishing the critical and non-critical attributes of each quadrilateral (i.e. having four equal sides is a critical attribute of a rhombus; however, having right angles is a non-critical attribute). Hershkowitz (1990) emphasized the need for using the critical properties of the concepts effective in decision making in learning. Later, the PSTs were asked to write definitions for each quadrilateral. Tall and Vinner (1981) argue that a concept definition generates its own concept image in learners' mind. Thus, creating opportunities for defining quadrilaterals as well as comparing various definitions were the focus of the instruction in week 2. Below is a sample definition that was proposed as a definition for defining a parallelogram by PSTs during class.

"A parallelogram is a quadrilateral with opposite sides parallel, equal and opposite angles are equal."
As it was evident in the definition above that the PSTs were not aware of the critical and non-critical attributes of a parallelogram. Instead, they chose to list most of the properties known to be a parallelogram. The importance of including minimal subsets of relevant attributes in concept definitions (Hershkowitz, Bruckheimer, & Vinner, 1987, p. 81) was discussed during the class. According to Hershkowitz (1990) the properties included in the definitions may form a basis for the decision making of individuals.

Research supports the idea of mental manipulation of shapes in order to understand hierarchical relations of quadrilaterals (Erez & Yerushalmy, 2006; Lehrer et al., 1998). However, Erez and Yerushalmy (2006) conjecture that in order to understand mental manipulation that preserves critical attributes learners must be aware of the existence of such attributes. Thus, critical and non-critical attributes of each quadrilateral were addressed again. Later, two definition types —hierarchical and partition— that were proposed by de Villiers (1994) were introduced and discussed. The PSTs were provided several definitions including hierarchical, partition, correct and incorrect and asked to decide (a) whether the definition is a working(correct) definition for the quadrilateral that is being asked to define (b) whether the definition is a hierarchical or partition definition (see Appendix B).

Week 3

The purpose of week 3 activities was to help PSTs strengthen their understanding on the properties of quadrilaterals and the relationships among them. First, the PSTs were asked to show the relationships between Trapezoids, Squares, Rectangles, Parallelograms, and Rhombi by using Venn Diagrams.The figure below (Figure 6) was one of the responses that were shared during instruction.
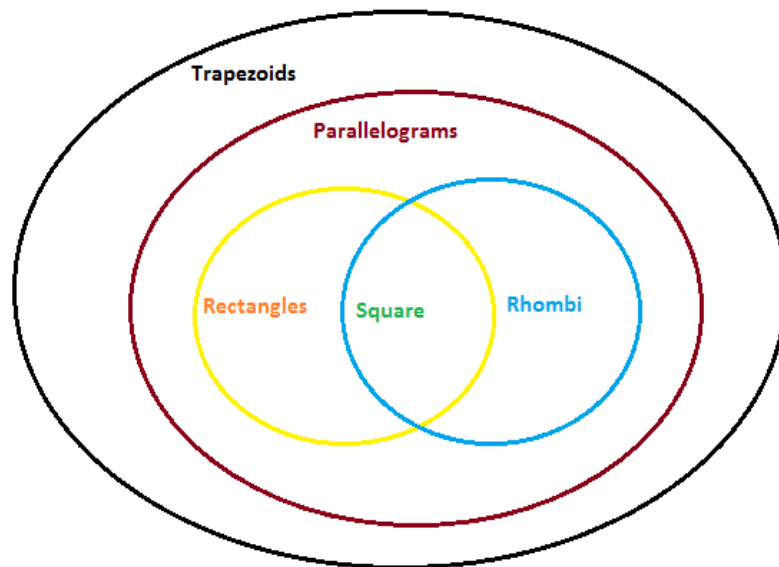

Figure 6. A sample response to the assignment question

Later, PSTs were given time to show the relationships between quadrilaterals including isosceles trapezoids and kites by constructing a concept map.
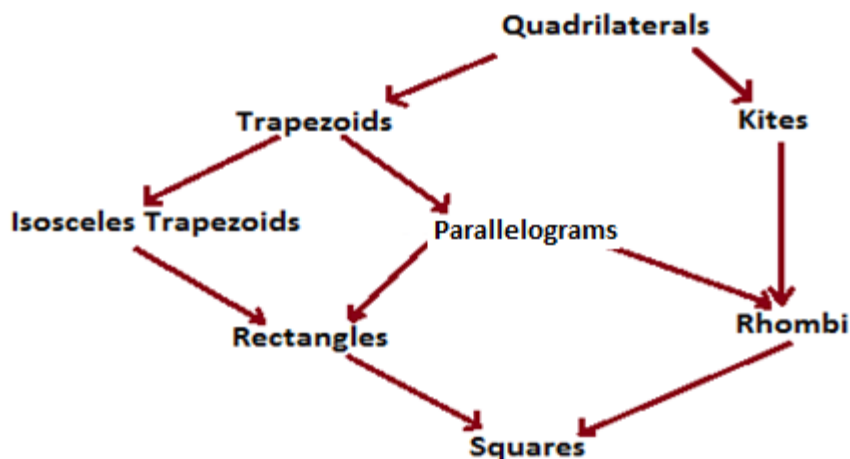
Figure 7. The concept map that was constructed during instruction

As a final activity of the week, the PSTs were challenged to incorporate Isosceles Trapezoids and Kites in their Venn diagram representation (Figure 6). Even though the PSTs completed the Venn diagram representation to show the relationships between trapezoids, parallelograms, rhombi, rectangles and squares previously, they struggled to incorporate isosceles trapezoids and kites in their previous representation. Below is a short excerpt to demonstrate how the PSTs struggled to incorporate new groups to their representations.

> Instructor: Tell me what you are discussing about?
> PSTC: We are discussing where we should put kites in this representation.
> Instructor: Ok, where do you think it should go?
> PSTC: Rhombi are also kites, so it should also include rhombi in it.
> PSTD: Also squares
> Instructor: Okay
> PSTs: Pause
> PST H: Okay, I think I know how we could do that. I think it should be like this (drawing Figure 8). Yes, so squares and rhombi are also kites so it should encapsulate both.
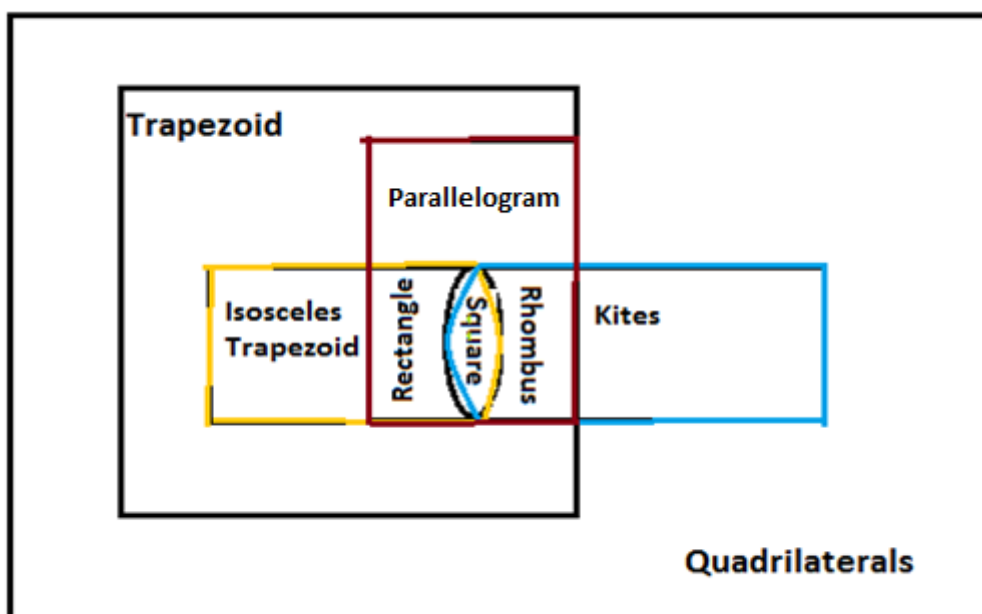

Figure 8. The PST H's diagram to show the relationships among quadrilaterals

As it was evident in the excerpt above, PSTs struggled to incorporate isosceles trapezoids and kites in the representation. However, it was also evident that the PSTs were thinking deeply about the inclusion relations of quadrilaterals in order to construct the representation.

Week 4

In this week, the activities focused on quadrilateral constructions with a compass and straightedge. The PSTs were first asked to construct a rhombus by only using a straightedge and a compass. They were encouraged to think about the properties of quadrilaterals to complete the constructions. Below (Figure 9) is a sample of a PST's way of constructing a rhombus.
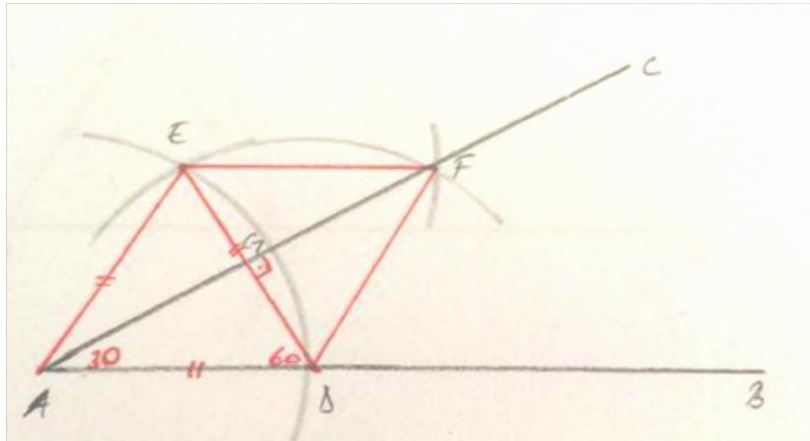
Figure 9. A sample of a rhombus construction

It was evident that the PST was able to construct two circles of the same radius to make sure of constructing four equal sizes in order to construct a rhombus. Additionally, the PST was not only able to mark the 90-degree angle between the diagonals of the rhombus, but also other interior angles by using the properties of polygons. It was clear in the construction that the PST was applying the critical attributes of a rhombus to his constructions.

After constructing a rhombus, the PSTs were asked to construct other quadrilaterals such as a square first and then asked to share their methods as a whole class. Below (Figure 10) is a sample of a PST's construction of a square. The PST used the definition of a square as a rhombus with a 90-degree angle. He argued that the constructed shape has four equal sides and one right angle. Therefore, it was a square.

Figure 10. Another sample of a square construction

Week 5

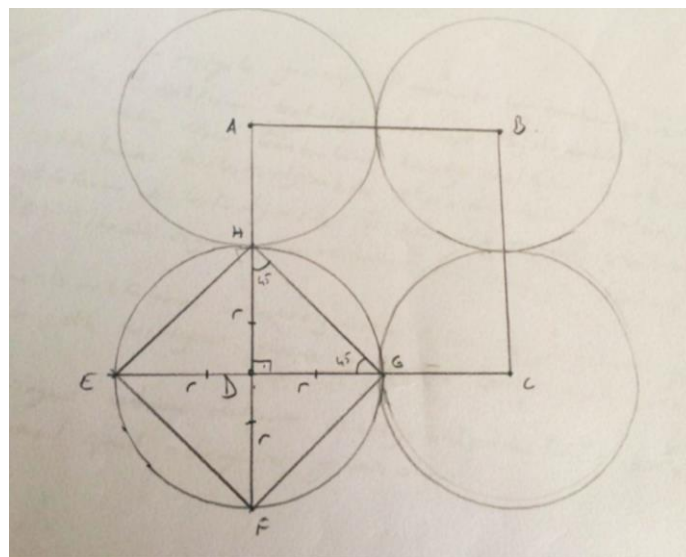The main purpose of this week was to introduce the van Hiele model of geometric thinking and discussing about the characteristics of each level in details. Then, the PSTs were engaged in a task to sort out some activities (adopted from van de Walle, 2001) according to the van Hiele levels.

Post Questionnaire Results

In the last week of the semester, PSTs were administered the same questionnaire that they were asked to complete in the first week of the semester. After analyzing their responses to the same questionnaire questions, it was evident that their conceptions of quadrilaterals have improved during the semester. As it was evident in the Table 2, the majority of the PSTs reasoned at hierarchical level.

During the pre-questionnaire administration, it was apparent that the majority of the PSTs struggled with trapezoids including what constitutes trapezoid and/or identifying quadrilaterals with at least one set of opposite parallel sides as a trapezoid. However, during the post questionnaire it was evident that all 48 PSTs' were clear about what constitutes a trapezoid. Additionally, they correctly identified a square, rectangle, rhombus or a parallelogram—a quadrilateral with at least one set of opposite parallel sides—as a trapezoid.
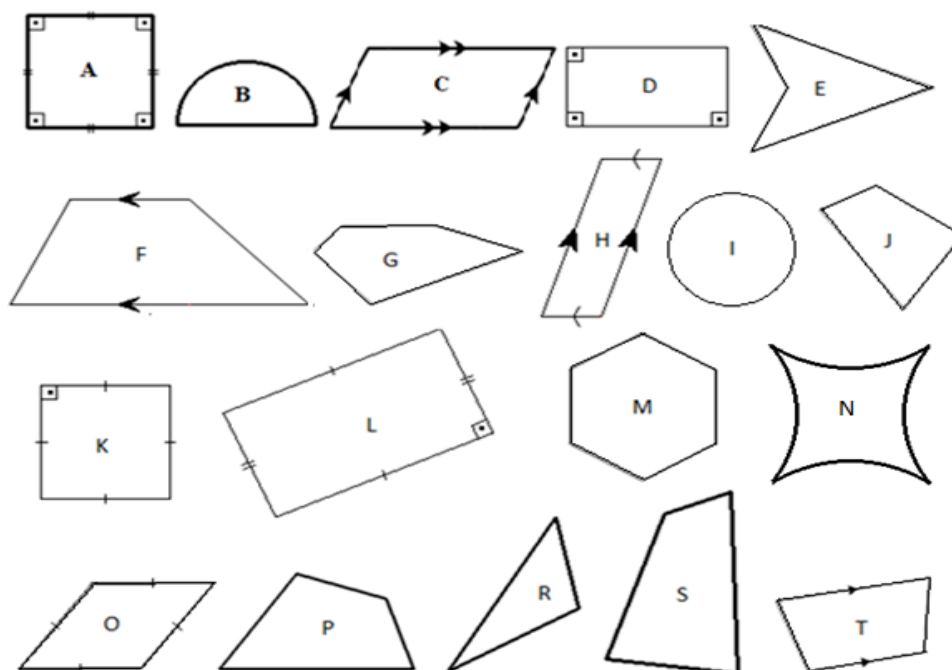
3- List all the shapes that you think is a **Trapezoid.**
    Answer: A, C, D, F, H, K, L, O, T
    Reason: A shape with at least two parallel sides is called a trapezoid

Figure 11. A sample response to the questionnaire item 1 in the post-questionnaire

Twenty-nine out of 48 PSTs were able to reason hierarchically in the post questionnaire. These 29 PSTs were not only able to reason inclusively about all quadrilaterals, but they were also able recognize properties as eligible for change, without losing example hood. For instance, these PSTs were able to recognize that changing the interior angles of a parallelogram to a ninety-degree angle would not result in loosing example-hood. 18 PSTs, on the other hand, still struggled with recognizing this dimensions of possible variation. In other words, they struggled recognize the features of some quadrilaterals as eligible for change, without losing example hood. These 18 PSTs held rather static view of quadrilaterals.

Fill in the blanks with "**Always**", "**Sometimes**", or "**Never**".

a) A square is.............*always*.............a trapezoid.

b) A rhombus is.............*always*.............a parallelogram.

c) A parallelogram is.............*sometimes*.............a rectangle.

d) A trapezoid is.............*sometimes*.............a parallelogram.

e) A rectangle is.............*sometimes*.............a square.

f) A square is.............*always*.............a rectangle.

g) A rectangle is.............*sometimes*.............a rhombus.

Figure 12. A sample response to a questionnaire item in the post-questionnaire

Conclusion and Discussion

This study aimed to document PSTs' understanding of inclusion relations of quadrilaterals and the changes in their understanding of quadrilaterals as a result of a designed quadrilateral unit. Although geometry is an integral part of the curriculum, many students fail to develop a deep understanding of basic geometric concepts (Clements 2003; O'Brien, 1999). According to the National Council of Teachers of Mathematics (NCTM), instruction should empower students with the ability to analyze properties of geometric shapes and to understanding of relationships among these properties from kindergarten through high school geometry (NCTM, 2000). Teachers play an essential role in the promotion of these skills and implementing these standards. However, it was stated in the previous studies, conducted with teachers and PSTs, that teachers demonstrated some of the similar struggles and misconceptions that were drawn from the studies conducted with students (Okazaki & Fujita, 2007; Türnüklü, Alaylı, &Akkaş, 2013). The findings of this study aligned well with the results of those studies by revealing that the PSTs in this study also struggled with identifying quadrilaterals—especially trapezoids—and understanding inclusion relationships among quadrilaterals at the beginning of the semester.

Okazaki (1995) suggests that some inclusion relations between quadrilaterals are easier to grasp for learners than others. The results of this study revealed that the PSTs were able to associate square-rhombus and rhombus-parallelogram easier than associating other quadrilaterals. Understanding square-rhombus or rhombus-parallelogram interrelations easier than inclusion relations between other quadrilaterals might be due to resemblances between their prototype shapes. Previous studies also document that parallelogram-rhombus association is one of the first interrelations that learners are able to understand while square-rhombus association is not quite easy to comprehend by the learners (Okazaki & Fujita, 2007; Türnüklü, 2014). However, this study documented that the PSTs were able to associate square-rhombus more easily than they could comprehend the inclusion relations between the other quadrilaterals. The results of this study also demonstrated that the PSTs struggled to understand the inclusion relations between trapezoids with squares, rectangles, parallelograms and rhombi the most.

Zazkis and Leikin (2007) believe that examples generated by participants mirror their conceptions of mathematical objects involved in an example generation task, their difficulties and possible inadequacies in their perceptions. Watson and Mason (2005) refer examples as "illustrations of concepts and principles". It was evident in the responses of the PSTs that their conceptions of trapezoids were limited. The majority of the PSTs were neither able to correctly identify examples of trapezoids nor the interrelations between trapezoids with other quadrilaterals in the pre-questionnaire results. Although the numbers were increased by the end of the semester, the PSTs still struggled seeing squares, rectangles, rhombi or parallelograms as subsets of trapezoids. Researchers argue that learners usually do not use definitions of concepts, even in the case when the concept does have a definition; rather they use concept images (Clements, 2003; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). The results of this study aligned well with the results of those studies by documenting that even

though the participants were able to correctly define trapezoids as quadrilaterals with at least one pair of parallel sides, when it came to identify examples they still selected the ones that looked like the prototypes of the shape.

Watson and Mason (2005) use the term dimensions of possible variation to refer to the features of an example that learners recognize as eligible for change, without losing example-hood. It was evident in this study that it is a complex thinking to develop among PSTs that some properties such as side lengths or angle measures are eligible for change without losing example-hood. Erez and Yerushalmy (2006) claim that understanding manipulation on a shape that preserves the critical attributes of the shape is necessary for constructing complete concept image. However, de Villiers (1994) claim that students often see figures in a static way rather than in the dynamic way that would be necessary to understand the inclusion relations of the geometrical figures. The findings of this study aligned with the result of these studies by documenting that the PSTs were not able to keep track of the visual changes and infer what had been changed and what had been preserved. Erez and Yerushalmy (2006) state that it is difficult to change the concept image of quadrilaterals and at the same time understand the mathematical logic of the mental manipulation on quadrilaterals. One reason might be the difficulty of tracking the visual changes and inferring what has been changed or what has been preserved when manipulating a shape (Erez & Yerushalmy, 2006).

Fujita's (2012) levels, formed for parallelograms, describe learners' cognitive development of understanding of inclusion relations among quadrilaterals. Although these levels provide an essential model to understand learners' understanding of inclusion relations of quadrilaterals, the levels seem to be too broad. The findings of this study demonstrated that the PSTs in Level 2 differed in terms of their understanding of inclusion relations. It was observed that while some PSTs in Level 2 had only extended their figural concepts of a couple of quadrilaterals (Level 2-c), some PSTs were able to understand the inclusion relations between most of the quadrilaterals (Level 2-a). Additionally, it was found that not all PSTs in this level were initially able to recognize transitive properties of quadrilaterals. Therefore, Level 2 was divided into further three subgroups according to similarities and differences of the PSTs' reasoning skills. Similarly, some of the PSTs held entirely prototypical reasoning, but extended their figural concept of one specific quadrilateral, which was either recognizing a square as a rhombus or a rhombus as a parallelogram. Since these four PSTs relied heavily on their prototypical examples for all quadrilaterals, except just for one specific quadrilateral, they were coded in Level 1-a. These PSTs were acknowledged as in transition to Level 2-c. Figure 13 below summarizes the groups and subgroups of understanding inclusion relations of quadrilaterals.

Figure 13. Levels of understanding inclusion relations of quadrilaterals

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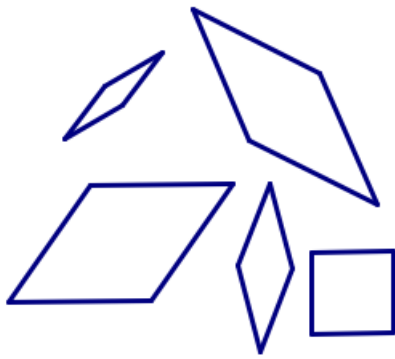
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Appendix A

Quadrilaterals
Properties of the Rhombus

Definition: A *rhombus* is a quadrilateral with all four sides
equal in length.

Use the given definition and polystrips, dotted paper or geo-boards to make a rhombus. Trace a rhombus on your paper and identify some of its properties. Try to identify additional properties. Be ready to share your findings!

Properties	List and Describe Properties of the Rhombus
Side Properties	
Angle Properties	
Diagonal Properties	
Symmetry Properties	

Appendix B
EVALUATE DEFINITIONS

Discuss the definitions provided below in your groups. Decide whether the definitions are correct or not first and then place them in the correct parts of the table. Be ready to share your thinking!

1. Rectangle is a quadrilateral with three right angles
2. Square is a rhombus with one right angle
3. Rectangle is a parallelogram with one right angle
4. Trapezoid is a quadrilateral with at least one pair of parallel sides
5. Parallelogram is a quadrilateral with opposite angles equal
6. Rectangle is a parallelogram with equal diagonals
7. Square is a rectangle with perpendicular diagonals
8. Parallelogram is a quadrilateral in which the diagonals bisect each other
9. Rectangle is a quadrilateral with all 90 degree angles
10. Parallelogram is a quadrilateral in which adjacent angles are supplementary
11. Square is a quadrilateral with all sides equal in length
12. Kite is a quadrilateral in which the diagonals bisect each other
13. Trapezoid is a quadrilateral with only one pair of opposite sides to be parallel
14. Parallelogram is a quadrilateral with opposite sides parallel
15. Square is a quadrilateral with all the angles and all the sides equal
16. Rhombus is a quadrilateral with opposite sides and angles equal
17. Kite is a quadrilateral with only two pairs of adjacent sides equal in length
18. Square is a rectangle with equal sides
19. Parallelogram is a quadrilateral with opposite angles equal and opposite sides to be parallel
20. Rectangle is a quadrilateral in which the diagonals bisect each other
21. Rhombus is a quadrilateral with opposite angles equal in measurement
22. Trapezoid is a quadrilateral with no symmetry axes
23. Square is a rhombus with equal diagonals
24. Kite is a quadrilateral with perpendicular diagonals

Correct Definitions	Incorrect Definitions	Economical Definitions	Non-Economical Definitions	Hierarchical Definitions	Partial Definitions