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Teaching the Concept of Limit by Using Conceptual Conflict Strategy and Desmos Graphing Calculator

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Abstract

Although the mathematics community has long accepted the concept of limit as the foundation of modern Calculus, the concept of limit itself has been marginalized in undergraduate Calculus education. In this paper, I analyze the strategy of conceptual conflict to teach the concept of limit with the aid of an online tool – Desmos graphing calculator. I also provide examples of how to use the strategy of conceptual conflict. This graphing calculator provides an interactive, dynamic, and persuasive approach of teaching limit. I focus on applying the conceptual conflict idea to the concept of limit in the situation where x approaches infinity. This strategy can be applied to the limit of a function when x approaches a fixed number.

Key words: Calculus; Conceptual change; Conceptual conflict; Limit; Technology

Introduction

Although the mathematics community has long accepted the concept of limit as the foundation of modern Calculus, the concept of limit itself has been marginalized in undergraduate Calculus education. Textbooks have paid little attention to its fundamental aspects – the $\delta - \varepsilon$ relationship and teachers treat this topic hurriedly and cannot wait to move to next topics (Bokhari & Yushau, 2006). For non-mathematics majors, lack of understanding of the concept of limit may not be a serious problem for their study of further mathematics courses; for mathematics majors, however, it is an important issue. Students' understandings of Calculus will greatly influence their ability to study more advanced Analysis courses (such as Real Variable Function, Real Analysis, Functional Analysis, and Measure Theory) because these courses all require Calculus as a prerequisite. There are many sources that contribute to the difficulties of teaching and learning this concept (Cornu, 1992; Davis & Vinner, 1986). Nevertheless, teaching the concept of limit successfully is not an unattainable task if we use proper strategies and tools. In this paper, I analyze the strategy of conceptual conflict for teaching the concept of limit with the aid of an online Desmos graphing calculator. I focus on applying the conceptual conflict idea to the concept of limit in the situation where x approaches infinity. This strategy can be easily applied to the limit of sequences and limit of a function when x approaches a fixed number. The idea of conceptual conflict can be generalized to other situations in Calculus, such as teaching the concepts of the continuity and derivative.

Theoretical Framework

A principal tenet of constructivist learning theory is that new knowledge (or concept) builds on prior knowledge (or concept) (Davis & Vinner, 1986; Hewson & Hewson, 1984; Meyer, 1993). *Conceptual conflict* is the conflict between the new concept with the learner's prior concept. The process of learning through connecting previous knowledge to new knowledge is often related to conceptual change. Vosniadou (2007) defines *conceptual change*:

> In order to understand the advanced scientific concepts of the various disciplines, students cannot rely on the simple memorization of facts or the enrichment of their naive, intuitive theories. They need to be able to restructure their prior knowledge which is based on everyday experience and lay culture, a restructuring that is known as conceptual change. (p. 47-48)

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When the connection between previous knowledge and new knowledge is unsuccessful, errors in understanding (or misconceptions) may appear (Meyer, 1993). For example, students may apply the experience of multiplication of positive integers to the multiplication of fractions so they may believe that multiplications always make numbers bigger. However, when students learn new knowledge they may not even be aware of the need to change the underlying logic of thinking and their own misconceptions (Merenluoto & Lehtinen, 2000). Although research on conceptual change is mainly conducted in the field of science education, there is increasing attention to this idea in the field of mathematics education (Vamvakoussi & Vosniadou, 2004).

Conceptual change requires students to see the necessities for a change in their concepts (Merenluoto & Lehtinen, 2000). If they do not recognize a need for conceptual change, they may develop misconceptions. For example, when students learn that the complex number system is an extension from real number systems, they may think that it is reasonable to compare any complex numbers because they can do that in the real number system. However, that is not the case for complex system. One cannot tell which one is bigger for any given two given imaginary numbers. In order to prevent and correct students' misconceptions, teachers should apply approaches to achieve students' conceptual change. In the fields of science and mathematics education, researchers have identified several useful strategies for supporting conceptual change: analogies, multi-representation, discussion, historical perspectives, and conceptual conflicts (Calik, Ayas, & Coll, 2008; Meyer, 1993; Nussbaum & Novick, 1982; Singer, 2007). It is reasonable to use different strategies in different situations. This paper will focus on applying the conceptual conflict approach to teaching the concept of limit.

According to Trumper (1997), if a conceptual change is needed for students, the first step is to let students be aware of the necessity of such a change and feel dissatisfaction with the explanation that is based on previous knowledge. Conceptual conflict is one of the effective ideas to help students to recognize such dissatisfaction (Meyer, 1993). If students fail to feel dissatisfaction, they are not likely to develop a conceptual change spontaneously. However, too often, students are not given the change to feel such dissatisfaction and are merely provided new knowledge by the teacher, new definitions and the classical experiments. According to Nussbaum and Novick (1982), "…students read about, observe or even perform those experiments before they are given the chance to recognize the existence and nature of the problem and they, therefore, do not get to experience any cognitive conflict at all" (p. 186). Although Nussbaum and Novick's (1982) article is mainly about science education, the situation is similar in mathematics education. In mathematics, students are taught by presentations of definitions and concepts rather than developing an understanding of why they need these definitions and concepts. This phenomenon is common in college level mathematics.

The strategy of conceptual conflict provides an approach for students to be aware of the need to make conceptual change (Nussbaum & Novick, 1982). The first step is to let students recognize explicitly their misconceptions--that is, the step of diagnosis. Instead of hastening to introduce new concepts to students, teachers can spend some time to identify students' previous understandings of the intended concepts because students often bring lots of informal mathematics knowledge to classrooms. Then teachers can provide some problems and ask students to solve and explain these problems in the class. Different students may have different solutions and explanations. Based on students' statements of their understanding, teachers will be able to categorize some features of their misconceptions.

At the second step, teachers can lead students to debate the pros and cons of these statements – that is, the step of clarification. After the debate and arguments of the students, some statements will survive, some will be eliminated, and some will be undecided. At this step, teachers will be able to summarize students' misconceptions: some of them may be particular to a few students and others may be common to many students.

At the third step teachers need to confront students' misconceptions. Teachers can try to find the logical problems within students' replies or present students with counterexamples that violate their misunderstandings. The counterexamples are expected to confront students' misconceptions explicitly. Teachers need to defend why a scientifically agreed-upon conception or a formal definition has more empirical or theoretical validity than students' conceptions. The basic tool is the logic: the formal concept should make sense. Some constructivists may argue it will be better to induce students to develop the new concepts by themselves (Bishop, 1967). But this is not always that case especially for the subjects like mathematics which has subsided for thousands of years with the endeavors of mathematicians. Students are necessarily able to develop the desirable mathematical understanding even if they feel dissatisfaction of their original explanation.

At the last step, it is hopefully that by now students are willing to accept the new knowledge and change their misconceptions – that is, the step of accommodation. At this step, teachers can help students to incorporate the

new knowledge to students' previous knowledge because teachers have created the necessary conditions to learn the new knowledge for students (Meyer, 1993).

Application of Conceptual Conflict to Teach the Concept of Limit

Why Limit Matters?

The concept of limit is situated in an ironic place in the current Calculus education. On one side, limit is the fundamental concept for modern Calculus and related subjects such as Measure Theory, Real Analysis, and Functional Analysis. Rarely do mathematicians refuse its fundamental role in Calculus and other Analysis. Students often begin to learn Calculus with the concepts of function and limit. If mathematics majors do not understand the concept of limit, they are not likely to understand the concepts of continuity, uniform continuity, convergence, derivative, and they are not likely be ready to take other Analysis courses. On the other side, the concept of limit has been marginalized in textbooks, teaching, and research. (Bokhari & Yushau, 2006). For example, textbooks usually pay little attention to the theoretical aspects of the concept of limit (Ostebee & Zorn, 2001; Stewart, 2012). Calculus teachers usually focus on the calculation of limit, sometimes on graphical illustration of limit, rarely on theoretical aspect (or definition) of limit.

One of the most important features that differentiate mathematics from other subjects is the upgrade from intuitive concrete understanding to abstract recognition. The concept of limit is such an example. Before students first learn the concept of limit, they already have some experiences of what a limit is (Merenluoto & Lehtinen, 2000). Their understanding is mainly based on everyday experiences rather than mathematical understandings. If students do not change their everyday understanding of limit to mathematical understanding of limit, they will not be able to upgrade from intuitive concrete understanding to abstract recognition. In order to reach the deep understanding of Calculus, curriculum and instrumental design need to be theoretically grounded in the actual mathematics and find an effective approach within the mathematics community. Despite of its importance, students find it difficult to understand the concept of limit. The next highlights the common sources of difficulties in teaching the concept of limit informed through the history of limit in Calculus.

Sources of Difficulties in the Teaching the Concept of Limit

There are many sources that may contribute to the difficulties of teaching the concept of limit. First, the difficulties may come from students' misconceptions of limit. Some researchers have identified some common misconceptions of limit: A sequence "must not reach its limit"; a limit is a boundary beyond which the sequence cannot go; a limit is a stop-sign, etc. (Davis & Vinner, 1986; Merenluoto & Lehtinen, 2000; Tall & Vinner, 1981). For example, the differences between the everyday language and the mathematics language contribute students' misconceptions (Cornu, 1992; Kim, Sfard, & Ferrini-Mundy, 2005). When students enter the Calculus classrooms they bring their everyday experiences and understanding of "limit" with them which though necessary can lead to misconceptions and hence also bring learning obstacles. For example, one may say that my limit of running is two miles, and three hours is my limit to keep continuous working. These everyday understandings of limit suggest that limit is some value one cannot go across.

Sometimes even Calculus book writers may be unable to notice these differences between everyday language and the mathematics language and thus add fuel to fire of students' misconception. For example, in order to explain the concept of limit to his or her student, one book use an example like this: in order to explain the meaning of $x \to 3$, the authors used an analogy: you can get access infinitely close to a running fan but obviously you will never reach it because you know what will happen if you reach the running fan (Adams, Thompson, & Hass, 2001). In this case, the book strengthened the misconception that limit is something that you can infinitely approach but never actually reach. The risk of this strategy is that when one person approach a fan, he usually only approach it from one side but $x \to 3$ mean $x \to 3^+$ and $x \to 3^-$. The lack of awareness of misconceptions of the concept of limit is one of the main sources of the current situation in both teaching and learning the concept of limit.

Second, students' perspectives on the learning of the concept of limit are another source of difficulty (Cornu, 1992; Szydlik, 2000; Tall & Vinner, 1981). Besides lack of recognition of misconception, most students also do not see the value of understanding the concept of limit: their teachers say this is optional; their textbooks place the definitions in the end of chapters; their experience tells them it is often useless in exams. Students and

teachers often use achievement in exams to judge whether or not students have mastered the intended knowledge. This perspective promotes students' belief that the most important thing about limit is to know how to calculate it instead of understanding it and how to prove the existence of limit. Meyer (1993) argued that students who rely exclusively on terminology, memory of text, or common sense may answer exam questions correctly, but they may not understand their answers fully or may still believe falsely (p. 106). Cornu (1992) confirmed that it looked like that lack of understanding the concept of limit is not a problem for students' exams.

> Different in investigations which have been carried out show only too clearly that the majority of students do not master the idea of limit, even at a more advanced stage of their studies. This does not prevent them from working out exercises, solving problems and succeeding in the examinations (Cornu, 1992, p. 154).

However, mathematic major Johnson's (2007) experience as indicates how lack of fundamental understanding of the concept of limit and convergence in Calculus hindered her study of Analysis.

Third, the difficulty in teaching the limit concept has a close relationship with purpose of the Calculus course, teachers' attitude, content knowledge and everyday instruction. Because typically both mathematics majors and non-mathematics majors take the same Calculus courses together, it is unlikely that the instructor would apply a rigorous treatment for this course. However, this may cause serious problem for mathematics majors thereafter.

On one hand, it is possible that some Calculus teachers themselves do not properly understand the concept of limit or have misconceptions of this concept. This is evident through the study by Mastorides and Zachariades' (2004) in which they indicated teachers' content knowledge of limit was incomplete and that it affected the pedagogical content knowledge. For example, "most of them have difficulties in understanding multi-quantified statements or fail to comprehend the modification of such statements brought about by changes in the order of the quantifiers (pp. 481-488)." Whether or not this happen to the college level Calculus teaching needs further investigation. On the other hand, some teachers may understand the concept of limit very well, but they are likely to think it is too challenging for students to learn so they do not emphasize this topic and just skip it.

> Teachers take the decision to present or exclude the $\varepsilon - \delta$ definition in their classes and some tell their students to memorize this definition; it will take them some time to absorb it. Many instructors put little emphasis on its explanation and look for an appropriate instance for its swift presentation before moving to the next topic. This may be one of the reasons that the definition does not hold a unified position in basic Calculus book (Bokhari and Yushau, 2006, p. 156).

Calculation and conceptual understanding are both important in the Calculus teaching. However, most emphasis has been placed on how to calculate the limit instead of on understanding its definition. Sometimes, proving the existence of limit is as important as finding the limit. For example, if students do not see the necessity to prove the existence of $\lim_{x \to \infty}(1 + \frac{1}{x})^x$ they are not likely to find its limit because e is actually defined by this limit!

When students take advanced Analysis classes they will feel a real need to understand the definitions of limit and the proofs of theorems related to the $\varepsilon - \delta$ way of thinking. This has been confirmed by Johnson (2007). She argued that when she took Calculus, she did not understand why it was necessary to prove that a sequence for which one has calculated a limit indeed has a limit. After she took analysis, she noticed that

> …the nature of this course (analysis) was different from ones I had taken in the past, but none of these differences were stated or clarified. The fact that the professor paid no attention to the differences in the nature of the mathematics courses made learning the material a difficult chore (Johnson, 2007, p. 286).

Fourth, textbooks' treatment of the concept of limit has a deep influence on teachers' instruction (Bokhari and Yushau, 2006; Cornu, 1992). Textbooks usually use "nice" and "good" examples when introducing new concepts (Gruenwald & Klymchuk, 2003). This holds for the concept of limit in particular (Gruenwald & Klymchuk, 2003). For example, at the sections about the limit of a function when x approaches infinity, two textbooks use the example $y = \frac{1}{x}$ for introduction (Ellis & Gulick, 2002; Ostebee & Zorn, 2001). The using of "nice" examples are pedagogically helpful but also risks of promoting students' misconceptions because it gives

student illusion that limit is something that a function can never reach. Davis and Vinner (1986) argued that the influence of specific examples may be a source of "naïve misconceptions" for understanding the concept of limit.

Many textbooks are inadequate at developing students understanding of the formal definition of limit which leads the learning of limit becoming mechanically applying procedures and techniques for calculation instead of conceptual understanding. For example, Ostebee and Zorn (2001), authors of the *Calculus: from graphical, numerical, and symbolic points of view*, spend more than four pages before the informal and formal definition of limits was given. Although the authors agree on the importance of limits, they treat the Calculus informally and students are not required to understand the meaning of its formal definitions. However, students who have used this textbook may plan to take advanced analysis courses but they do not know that they are not actually ready for these courses, this can be confirmed by Johnson's (2007) experience. Here are Ostebee and Zorn's (2001) comments in their textbooks:

> With a satisfactory theory of limit, based on precise definitions, mathematicians can makes sense of – and calculate reliably with – mysterious-seeming concepts such as "infinitely small quantities". Without such theory as footing, we would remain mired (as mathematicians did for hundreds of years) in the vague, metaphorical – sometimes even mysterical – language of "fluxions," "fluents," and "indivisibles" (Ostebee and Zorn, 2001, p. 152).

Lastly, there are the inborn deficiencies of the definition of the concept of limit: ambiguity of the $\varepsilon - \delta$ expression; infertile of the definition for calculation the limit. The multi-quantified statements such as for "any" and "there exist" strengthen the difficulty. Some students see it is illogic to prove a given limit (Johnson, 2007). Some students do not have a concrete feel of the existence of the δ (or M) and do not know how to find them. For most teachers and students, Weierstrass' definition of the concept of limit serves as more of a theoretical implication than having practical meaning, or they just think it is unnecessary to understand it. Weierstrass' definition of limit is useful for the theoretical construction of the building of Calculus, but it is weak in promoting cognitive understanding. It seems of little help for application to practical problems. It provides a method to prove a function or sequence's limit when one already knows the limit but it tells nothing about how to find the limit.

Strategy of Teaching the Concept of Limit

In order to enhance the quality of teaching the concept of limit, some researchers have put forward different strategies. For example, some researchers have suggested alternative representations to define the concept of limit (Bokhari & Yushau, 2006; Lynch, 2000; Todorov, 2001). However, in the near future, until now there is no clue that mathematics community will abandon Weierstrass' definition and adopt new definitions widely. Considering the fact that regardless of its difficulties, Weierstrass' definitions have been accepted by mathematics community for more than a century, it is not wise to abandon it. Furthermore, it is unlikely that there will be a drastic change in the atmosphere of Calculus textbooks to treat the position concept of limit. Instead some researchers suggest applying new approaches to teach the concept of limit – the Weierstrass' definition. For example, Cheng and Leung (2015) have studied how to use a dynamic applet to teach the concept of limit of a sequence. The method I suggest is using conceptual conflict with the aid of an online Desmos graphing calculator to teach the concept of limit of a function when x approaches infinity. The strategy of applying conceptual conflict in education, especially in science and mathematics education is not a new idea (Piaget, 1978). Klymchuk (2010) published a book about using counterexamples in calculus. However, his books and his other articles (Gruenwald & Klymchuk, 2003; Klymchuk, 2005) neither discussed about how to identify students' misconceptions nor talked about how to use the idea of conceptual conflict to rationalize the necessity of introducing new concepts.

At first, because many students come to classrooms with misconception or everyday language and experiences of limit, it is necessary for teachers to help students diagnose their misconceptions. I suggest using conceptual conflict method to let them know the difference between everyday language of limit and the mathematics language of limit. Students' experience-based knowledge of limit is correct in their every life but if they bring these understandings directly to the field of mathematics that will cause misconceptions. Based on their daily experience students often believe that limit is something you can approach infinitely but never actually reach. Just as I have mentioned in above, the identified some common misconceptions of limit include: A sequence "must not reach its limit"; a limit is a boundary beyond which the sequence cannot go; a limit is a stop-sign, etc. (Davis & Vinner, 1986; Merenluoto & Lehtinen, 2000; Tall & Vinner, 1981). In order to diagnose students'

misconceptions of limit, at the beginning of the class, the teacher can ask students to state what their understandings of limit are.

Second, students may have similar, different and even opposite perspectives of limit. In order to clarify students' understanding of limit, the teacher can ask them to debate their own statements of limit. The whole class should try to find some common features of students' statements of limit at this stage. Hopefully the teacher will get similar misconceptions of limit from students as discovered by other researchers.

Third, in order to produce conceptual conflict, I suggest starting with a counterexample $\lim\limits_{x \to +\infty} \dfrac{\sin x}{x} = 0$ to confront students' misconception explicitly. After graphing the function $f(x) = \dfrac{\sin x}{x}$ with Desmos graphing calculator, students will see that when x approaches infinity the value of the function $f(x) = \dfrac{\sin x}{x}$ will be iteratively larger, equal, and smaller than zero – the limit of $f(x) = \dfrac{\sin x}{x}$ as x approaches infinity. The detailed strategy will be addressed in the next section. Students' awareness of their misconceptions will initialize their need to take conceptual change although this step of diagnosis is not sufficient for students to make conceptual change. This dissatisfaction of their everyday explanation of limit will force students to seek a mathematically rigorous definition of limit. Students will be asked to find such a definition by themselves. However, I do not anticipate them to provide a mathematically adequate definition but their attempts will be cognitively helpful.

Lastly, the teacher provides the formal definition of limit, make remarks of the definitions. Examples of how to use the definition will follow. At this step, students will be able to make at least partial accommodations to their misconceptions.

What is Desmos graphing calculator?

Desmos graphing calculator can be accessed by users online either on laptops, desktops or mobile phones (available on Apple Store and Google Play). It allows users to graph multiple functions on the same screen. The output of the graph is very concise. As user can edit the commands conveniently and navigate on the screen freely it enables an interactive illustration of graphs for the user. User can also save the graph. When graphs are created user can copy the website's link and share it with other people and they will be able to see the command and graph and they can edit the commands. Desmos graphing calculator's capacity as an advance graphing calculator is further strengthened by the fact that it is free and user-friendly. In the following section, I describe an example lesson to use the idea of conceptual conflict and Desmos graphing calculator to teach the concept of limit.

Example of Teaching the Concept of Limit

This lesson aims to help students be aware of their misconceptions of limit, especially those caused by the difference between their everyday understanding of limit and the mathematical meaning of limit. Students will learn to see the need to develop a formal definition of limit different than everyday language of limit. Students will understand why we need limit as the foundation of Calculus. Students will understand the $\delta - M$ relationship and be able to prove simple problem in the case when x approaches positive infinity.

This instructor will use the idea of conceptual conflict to teach the concept of limit aided by online computer software Desmos graphing calculator. The first endeavor of the lesson is to use conceptual conflict strategy to make students feel a need of change of their everyday understanding of limit in order to respond to the dissonance in the example. The second strategy of this lesson is to treat limit as a dynamic mathematics conception. This method will give student a dynamic image of how the concept of limit could look like.

Before teaching limit of a function, students are expected to have learned the chapter of function. In the following paragraphs, I will illustrate the lesson process of teaching the concept of limit.

Step 1: Diagnosis of misconceptions – introduce the topic of "limit".

Based on literature review (Davis & Vinner, 1986; Merenluoto & Lehtinen, 2000; Tall & Vinner, 1981), there are some common misconceptions of limit: A sequence "must not reach its limit"; a limit is a boundary beyond which the sequence cannot go; a limit is a stop-sign, etc. Although these misconceptions have different presentations, they can be categorized into one group: limit is bounded; the sequence (or function) cannot reach it and they must "stop" before reach it. The teacher asks students to make mathematical examples that have the word "limit" in it. Students may have some answers like the "my limit of reading is 400 words per minutes." It is possible there are some other misconceptions that the current literature has not yet discovered. Students are required to write their answers on papers and read them out loud in the class.

Step 2: Clarification of students' conceptions.

The teacher (or voluntary students) writes down students typical answers on blackboard. Students need to explain explicitly what their understandings of the word "limit" are. The teacher asks students to argue and debate their understandings of limit. According to students' answers the teacher will be able to categorize some features of their understanding of limit. According to the literature (Davis & Vinner, 1986; Merenluoto & Lehtinen, 2000; Tall & Vinner, 1981), the class will get the following common features of students' conceptions of limit: limit is something you can get closer and closer to but never reachable; limit is a boundary (either upper boundary or lower boundary) that the sequence of function cannot go across.

Step 3: Confrontation of students' misconceptions – the use of counterexample.

Example 1. Students will be asked to work in groups to find the limit of $f(x) = \dfrac{\sin x}{x}$ when x approaches positive infinity. They can use any calculators if they want. Based on calculation, students may find that the overall tendency of the value $f(x)$ is towards zero.

The teacher asks students to explain why they think that $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$. Base on literature, some students may reply that it is because when x becomes larger and larger $\dfrac{\sin x}{x}$ becomes smaller and smaller. Then the teacher can say if $x_1 = 2\pi$, $f(x_1) = \dfrac{\sin x_1}{x_1} = \dfrac{\sin(2\pi)}{2\pi} = 0$; if $x_2 = 2\pi + \dfrac{\pi}{2}$,

$f(x_2) = \dfrac{\sin x_2}{x_2} = \dfrac{\sin(2\pi + \frac{\pi}{2})}{2\pi + \frac{\pi}{2}} = \dfrac{2}{5\pi}$. As a result, although $x_2 > x_1$, but $f(x_2) > f(x_1)$, that is when x becomes larger, $f(x)$ becomes larger in this case. This is a contradiction to the claim that "when x becomes larger and larger $\dfrac{\sin x}{x}$ becomes smaller and smaller."

If students think that $f(x)$ cannot reach zero, the teacher can ask them to calculate the values of $f(x)$ when $x = n\pi$, $n = 0,1,2,\cdots$. This will be conflict with their conception that limit is something never the sequence of function can reach.

Then the teacher asks students to answer whether or not the value of $f(x)$ can be greater or smaller than zero as x approaches infinity. If they discover that $f(x)$ can be both positive and negative during the process as x approaches positive infinity that will be conflict with their conceptions that limit is a boundary that a sequence or function cannot go across.

Then the teacher will ask students to sketch the graph of this function. Finally, the teacher will graph the function with the Desmos graphing calculator. As we seen in graph 1, the graph of $f(x)$ goes iteratively above and below the x-axis as x approaches positive infinity. The graph of the function looks like a wave with swing becoming weaker and weaker periodically but the swing never vanish. So although the limit of $f(x)$ is zero as x approaches positive infinity, the value of $f(x)$ can be equal, greater or less than zero. This will be a conceptual conflict of students' misconception that limit is unreachable, a boundary, or a stop-sign.
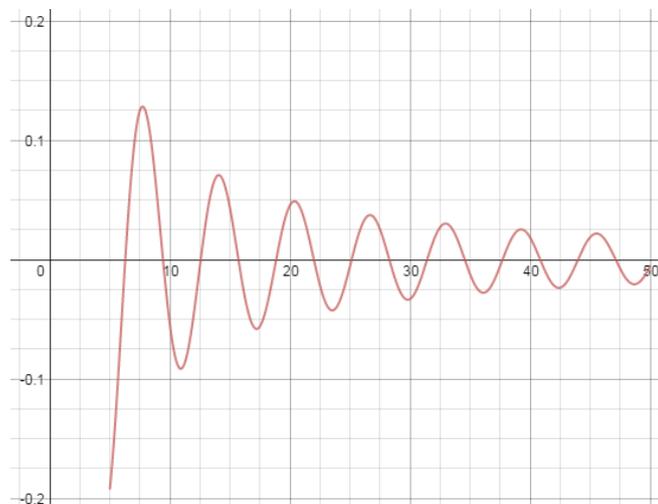
Graph 1. $f(x) = \dfrac{\sin x}{x}$

This conceptual conflict will lead to students' dissatisfaction of their previous understanding of limit. Then the teacher will ask students to make accommodation to the understanding of limit. However, it is not realistic to expect them to provide a precise mathematical definition of limit. That is nearly impossible because mathematicians took more than a century to reach the rigorous foundation of Calculus which based on the concept of limit. The achievement of this conceptual conflict is to let students be aware their misconception and feel the need to develop a precise definition for the conception of limit.

Step 4: Accommodation

The conceptual conflict in the above example will force students to abandon the closer and closer, larger and larger, or smaller and smaller expressions of limit because as we have seen, the function can be closer, equal and distant from the limit as x approaches positive infinity. So something different must be applied to define the definition of limit. Then the teacher will ask students to think of other ways to describe limit. However, the teacher should not anticipate getting a mathematically rigorous candidate from students' answers.

Then the teacher will suggest the use of distance as a measure of "closeness", that is, if the limit of some function is L, as x gets "large enough", say x_0, then the distance between $f(x_0)$ and L should be "small enough". Students are expected to know to use the absolute notion to describe the distance of two numbers if we do not know which one is larger. However, the problem is that how to describe mathematically "large enough" and "small enough". At this point, the teacher will provide students with the formal definition of limit, that is Weierstrass' $\delta - \varepsilon$ (or $\delta - M$) definitions.

Definition 1 (formal): Let a function f be defined on an infinite interval (a, ∞) for a real number a, and let L be a real number. The statement $\lim_{x \to \infty} f(x) = L$ means that for every $\varepsilon > 0$, there is a number $M > 0$ such that if $x > M$, then $| f(x) - L | < \varepsilon$.

There are two remarks about the definition. Remark 1: The definition provides one formal meaning of limit instead of telling us how to calculate what it is. We can use this definition to prove the limit of $f(x)$ is L.

During the proof of $\lim_{x \to \infty} f(x) = L$, the importance is the existence of such a M rather than of what it is. Because ε can be any given small positive number, it is not sufficient to prove $\lim_{x \to \infty} f(x) = L$ just for some specific ε 's.

Remark 2 (geometric explanation): At first, we draw a graph $y = f(x)$. For any $\varepsilon > 0$, $y_1 = L + \varepsilon$, and $y_2 = L - \varepsilon$ are a pair of line parallel to x-axis which composite a strip area S with $y = L$ in the central and with wide 2ε. In definition, "if $x > M$, then $| f(x) - L | < \varepsilon$" means that on the right-side of $x = M$ curve $y = f(x)$ will be completely restricted on the areas S. If we have smaller ε, then we will have narrower strip area S (because 2ε will be smaller as well), then the line $x = M$ is likely to move on the right side (because usually M will be bigger). However, no matter how narrow the strip area S is, there always exists a positive real number M such that on the right of $x = M$, curve $y = f(x)$ will be totally restricted in the strip area S.

Example 2. Given $f(x) = \dfrac{\sin x}{x} + 3$, we want find when (that is, to find M) the distance between $f(x)$ and 3 is less than 0.1, that is $| f(x) - 3 | < 0.1$ or equivalently, $2.9 < f(x) < 3.1$. Using Desmos graphing calculator, we can see a strip area S with wide 0.2, and we can choose M to be 20 (based on observing the graph 2).
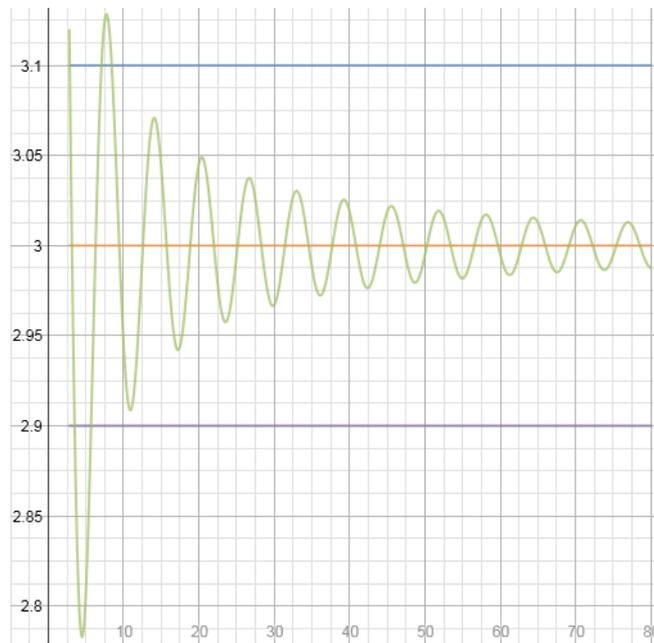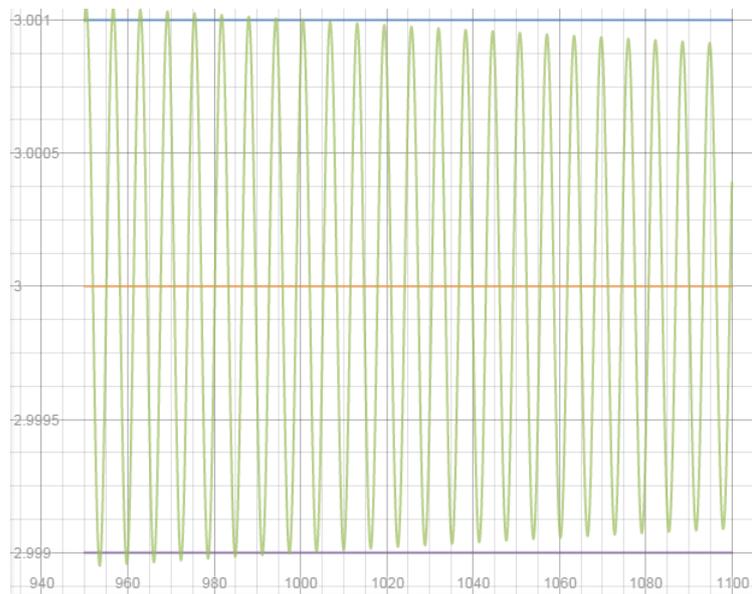

Graph 2. $f(x) = \dfrac{\sin x}{x} + 3$, $\varepsilon = 0.1$

The teacher will ask students to identify how close they like the $f(x)$ and 3 to be. Say, students want the distance between $f(x) = \dfrac{\sin x}{x} + 3$ and 3 is less than 0.01. Similarly, we have the strip area S with wide 0.02. If we can let M to be 110 that should be enough to guarantee $f(x)$ located in the strip area (based on observation the graph 3).
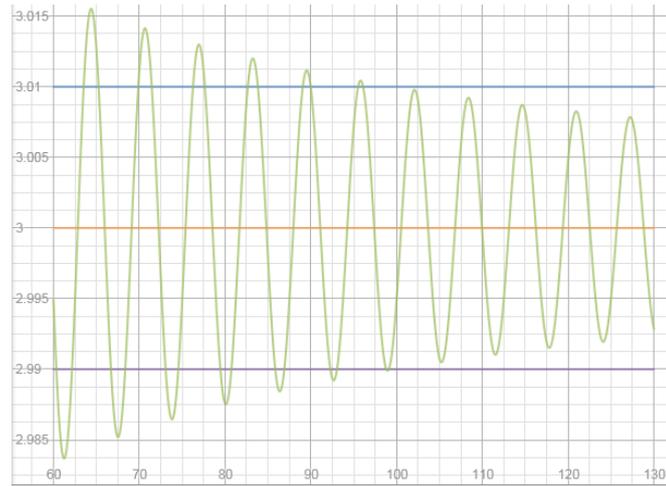
Graph 3. $f(x) = \dfrac{\sin x}{x} + 3$, $\varepsilon = 0.01$

This interactive exploration between teacher and students can be extended easily to other situation with the aid of Desmos graphing calculator. For instance, if some students want the distance between $f(x) = \dfrac{\sin x}{x} + 3$ and 3 is less than 0.001, there is a strip area S with wide 0.002. If we can let M to be 1040 that should be enough to guarantee $f(x)$ located in the strip area (based on observation the graph 4).

Graph 4. $f(x) = \dfrac{\sin x}{x} + 3$, $\varepsilon = 0.1$

In conclusion, no matter how close the students want the function to be with 3 (that is, there will be a narrow strip), the teacher can always find a "safe" positive number M such that after this point, the graph of the function locates within the narrow strip the students previously required. This dynamic process is very powerful and persuasive as it allows students to name whatever distances they want the function to be with a fixed number (its limit when x approaches positive infinity) and the teacher can always give the students a good answer (find the number M). Students should now have a hunch that no matter how small (the ε) they want the distance to be, they can also find a M to meet the requirement.

With the aid of Desmos graphing calculator, we have dynamically shown that we can find M's for $\varepsilon = 0.1$, $\varepsilon = 0.01$, or 0.001. But this is not a sufficient prove because in the definition we need to show the existence of M for any given ε. Then the teacher can teach students how to find the M for any given ε, that is, to prove $\lim_{x \to \infty} f(x) = L$.

Example 3. Prove $\lim_{x \to +\infty} f(x) = \dfrac{\sin x}{x} + 3 = 3$

Proof: For any ε, if we want $|\dfrac{\sin x}{x} + 3 - 3| = |\dfrac{\sin x}{x}| < \dfrac{1}{x} < \varepsilon$, we must have $x > \dfrac{1}{\varepsilon}$. So we let $M = \dfrac{1}{\varepsilon}$,

then when $x > M$, we have $|\dfrac{\sin x}{x} + 3 - 3| = |\dfrac{\sin x}{x}| < \dfrac{1}{x} < \dfrac{1}{M} = \varepsilon$. Hence, $\lim_{x \to +\infty} f(x) = \dfrac{\sin x}{x} + 3 = 3$.

In the proof the limit of function, the general method is to first assume that have $|f(x) - L| < \varepsilon$, then we need to find out what should M be based on the expression $|f(x) - L| < \varepsilon$. After the proof, the teacher can show that the observed value of M can be explained by the formal proof. For example, when $\varepsilon = 0.01$, they chose 120 based on observation. According to the proof it is enough to have $M = \dfrac{1}{0.01} = 100$.

Example 4: Prove $\lim_{x \to +\infty} \dfrac{2x^2 - 3}{x^2 - 4} = 2$ and provide graphic explanation.

Proof: Because we consider the situation when x approaches positive infinity, we can let $x > 2$. Then $x^2 - 4 > 0$. So if we want $|\dfrac{2x^2 - 3}{x^2 - 4} - 2| = |\dfrac{2x^2 - 3 - 2x^2 + 8}{x^2 - 4}| = |\dfrac{5}{x^2 - 4}| = \dfrac{5}{x^2 - 4} < \varepsilon$, we have

$x > \sqrt{\dfrac{5}{\varepsilon} + 4}$. Then we let $M = \max(2, \sqrt{\dfrac{5}{\varepsilon} + 4})$, when $x > M$, we have $|\dfrac{2x^2 - 3}{x^2 - 4} - 2| < \varepsilon$. $\square$

Graphic explanation: Let $\varepsilon = 0.001$, then the two horizontal boundary lines are $y = 2 + 0.001 = 2.001$ and $y = 2 - 0.001 = 1.999$. Let's check the graph for x ranges from 20 to 400.
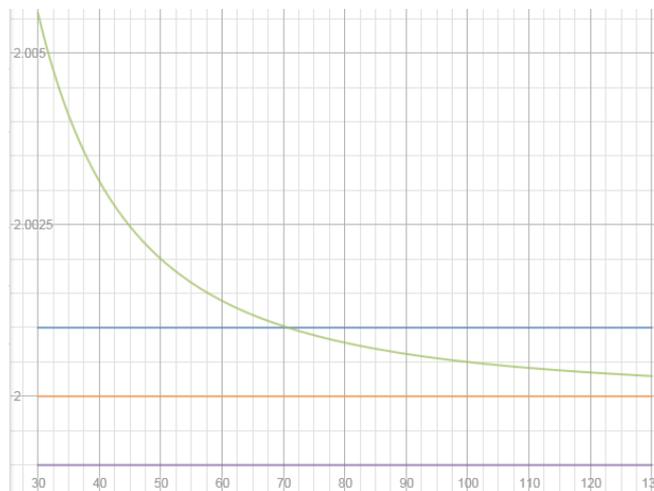

Graph 5. $f(x) = \dfrac{2x^2 - 3}{x^2 - 4}$, $\varepsilon = 0.001$

Numerical check: in the graph, we set $\varepsilon = 0.001$, then $\sqrt{\dfrac{5}{\varepsilon} + 4} = \sqrt{\dfrac{5}{0.001} + 4} = 70.74$.

So $M = \max(2, \sqrt{\dfrac{5}{\varepsilon} + 4}) = 70.74$. Hence the distance between the graph of $f(x) = \dfrac{2x^2 - 3}{x^2 - 4}$ and 2 is less than 0.001 for any x larger than 70.74. This is true according to our graph.

The teaching of limit usually include: the limit of a function when x approaches infinity, that is, $\lim\limits_{x \to \infty} f(x) = L$; the limit of a function when x approaches a fixed number x_0, that is, $\lim\limits_{x \to x_0} f(x) = L$; the properties and theorems of limit. This paper focuses on the part $\lim\limits_{x \to \infty} f(x) = L$, specifically $\lim\limits_{x \to \infty^+} f(x) = L$. However, similar teaching strategies will be applied to the rest of this topic.
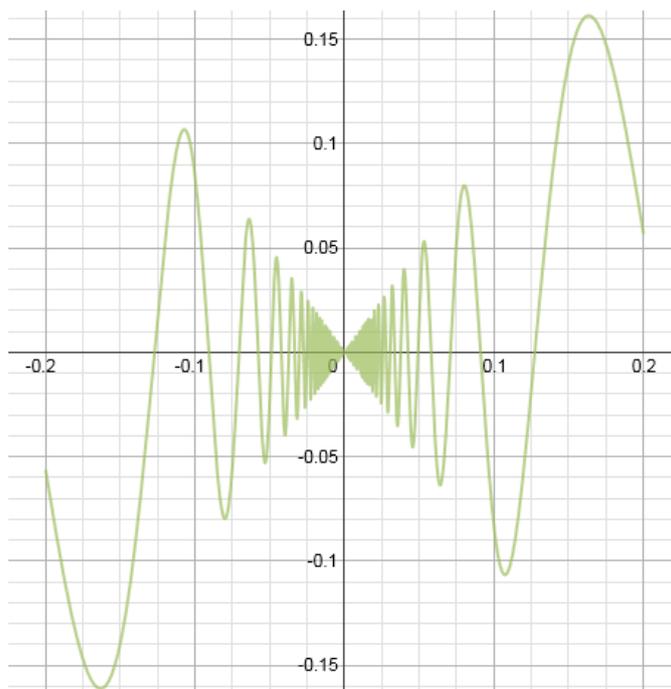
Graph 6. $f(x) = x \cos(\dfrac{1}{x})$

For example, another example $\lim\limits_{x \to 0} x \cos(\dfrac{1}{x}) = 0$ can illustrate the fact that when x approaches 0, the function $f(x) = x \cos(\dfrac{1}{x})$ can be equal, larger and less than the limit (see Graph 6).

In this article I have designed a strategy to apply the idea of conceptual conflict as a tool to launch students' conceptual change of the concept of limit. There are several differences between my strategies of teaching the concept of limit and other approaches: first, I pay attention to students' misconceptions of limit before teach it – that is the steps of diagnosis and clarification, and later confront students' miscomputations; furthermore, the lesson starts with a less "nice" examples, e.g., use $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$ instead of $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$; third, the lesson uses mathematics software to provide students a dynamic visualization of functions. This technique helps students to "see" the existence of the M for any given ε. Lastly, the requirement of conceptual understanding will be strengthened by the requirement of mastering the reasoning and techniques of proving limit.

Discussion

The need of students' conceptual change has been long under-addressed in education (Vosniadou, 2007). Meyer (1993) argued that "Teachers should have two goals: to teach the content, and to teach the necessary conditions for learning it (p. 106)." However, on one hand, teachers often do not provide enough attention to the necessary conditions for learning the content. Students are given few opportunities to feel any dissatisfaction of their previous knowledge. Too often, students are taught the content with little awareness of their own misconceptions and the reasoning behind the content. Students do not have opportunities to experience the conflict between their own misconceptions and the formal understandings and reasoning held and practiced by disciplinary experts. On the other hand, most current college Calculus textbooks are written with non-theoretical, informal, graphical, numerical, and symbolic perspectives which are not rigorous enough for mathematics majors. I do not mean to underestimate the value of textbooks with these perspectives. However, undergraduate Calculus classes usually have both mathematics majors and a large proportion of non-mathematics majors. These classes emphasize calculation more than proving and conceptual understanding, which may result in mathematics majors being unprepared for advanced Analysis courses. Calculus textbooks treat this subject in the calculation and procedure application view rather than theoretical perspective. As a result, students become passive receivers instead of critical thinkers; mathematics learning becomes a process filled with rote and applying calculation procedures. Mathematics majors are not ready for advanced Analysis courses. Johnson (2007) provided an example about how she (mathematics major) felt unprepared for Analysis course after she finished her Calculus course.

I argue that instead of retreating from teaching students with fundamental college level mathematics, mathematics teachers and textbooks writer should be confident in the learning ability of students and apply new approaches to teach formal and rigorous mathematics. By applying the approach of conceptual conflict to college level mathematics studies, this article provides an example of how to apply alternative strategies to teacher college level mathematics. The conceptual conflict can be applied to many other concepts in Calculus, such as continuity and derivative. Teachers may use different strategies at different situations. However, the issue is that when teachers teach mathematics to students, they should also teach the underlying logic and reasoning of the mathematics rather than a sole focus on procedures or calculations.

Desmos graphing calculator plays an important role in the implementation of the conceptual conflict strategy teaching of limit. However, the employment of technology is not a replacement of mathematic logic and reasoning. Instead, technology is as an auxiliary approach for students' mathematics learning. The requirement of critical and logic thinking, mathematics reasoning and proof should not be reduced. Students still need to learn how to do mathematics proof in the traditional way which is accepted by the mathematics community.

There are several limitations of the conceptual conflict strategy with the aid of Desmos graphing calculator. First, conceptual conflict does not necessarily lead to conceptual change because rejecting an old way of thinking may be difficult, no matter how convincing the new evidence seems (Meyer, 1993; Watson & Konicek, 1990). Second, the main goal of conceptual conflict was to let students be aware of their misconceptions instead of revolutionarily reconstructing students' way of thinking (Trumper, 1997). In fact, other approaches such as multi-representations and analogies also cannot guarantee a fundamental conceptual change for students. All these methods can do is to improve the likelihood of conceptual change instead of actually leading to conceptual change. Third, the idea of conceptual conflict cannot be universally applied to all teaching settings. It may work well in some cases but not in other cases. Hewson and Hewson (1984) agreed that not all learning needs to be conflict-induced. For example, when teaching the concept of quadratic function, using graphic representations will be a good method. Fourth, usually the strategy of conceptual conflict takes longer than most traditional teaching method and make it is difficult to expand to broad situation considering the limited teaching time for most teachers.

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